A Framework for the Natures of Negativity in Introductory Physics

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Positive and negative quantities are ubiquitous in physics, and the sign carries important and varied meanings. Unlike physics experts, novices struggle to understand the many roles signed numbers can play in physics contexts, and recent evidence shows that unresolved struggle carries over to subsequent physics courses. The mathematics education research literature documents the cognitive challenge of conceptualizing negative numbers as mathematical objects. We contribute to the growing body of research that focuses on student reasoning in a physics context about signed quantities and the role of the negative sign. This paper contributes a framework for categorizing the natures of the negative sign in physics contexts, inspired by the research into the natures of negativity in algebra. Using the framework, we analyze several published studies associated with reasoning about negativity drawn from the physics education and mathematics education research communities. We provide implications for mathematics and physics instruction and further research.

Keywords: Negative, Quantity, Negativity, Physics, Signed

Introduction

Development of mathematical reasoning skills is an important goal in introductory physics courses, particularly those geared toward students majoring in physics and engineering fields. Positive and negative quantities are ubiquitous in physics, and sign carries important and varied meanings. Unlike physics experts, novices struggle to understand the many roles sign plays in physics contexts.

Negative pure numbers represent a more cognitively difficult mathematical object than positive pure numbers do for pre-college students (Bishop et al., 2014). Mathematics education researchers have isolated a variety of natures of negativity fundamental to algebraic reasoning in the context of high school algebra that go beyond a ‘position on a number line’ nature (Gallardo & Rojano, 1994; Nunes, 1993; Thompson & Dreyfus, 1988). These various natures of negativity form the foundation for scientific quantification, where the mathematical properties of negative numbers are a good representation of natural processes and quantities. Physics education researchers report that the majority of calculus-based physics students struggle to make meaning of positive and negative quantities in spite of successfully passing Calculus I and beyond in mathematics (Brahmia & Boudreaux, 2016, 2017). Developing flexibility with negative numbers is a known challenge in mathematics education, and there is mounting evidence that reasoning about negative quantity poses a significant hurtle for physics students at the introductory level and beyond.

Few published studies have focused on negativity in the context of the mathematics used in physics courses. Studies conducted in the context of upper division physics courses reveal robust student difficulties (Hayes & Wittmann, 2010; Huynh & Sayre, 2018). Brahmia and Boudreaux constructed physics assessment items based on the natures of negativity from mathematics education research (Vlassis, 2004) and administered them to introductory physics students in the introductory sequence of courses (Brahmia, 2017; Brahmia & Boudreaux, 2016, 2017). The authors

Table 1: A map of the different uses of the negative sign in elementary algebra (Vlassis, 2004)

Unary (Struct. signifier)	Symmetrical (Oper. signifier)	Binary (Oper. signifier)
Subtrahend	Taking opposite of,	Completing
Relative number	or inverting the operation	Taking away
Isolated number		Difference between numbers
Formal concept of neg. number		Movement on number line

report that students struggle to reason about signed quantity in the contexts of negativity typically found in the curriculum (e.g., negative work, negative acceleration in one dimension, negative direction of electric field), and they concluded that science contexts may overwhelm some students' conceptual facility with negativity. In addition, they observed that students struggled to interpret the meaning of either a positive or negative signed quantity—it is the existence of a sign that causes difficulty (Brahmia & Boudreaux, 2017). These studies reveal that signed quantities, and their various meanings in introductory physics, present cognitive difficulties for students that many don't reconcile before completing the introductory sequence.

The current study contributes to this body of research by introducing a framework for categorizing the natures of negativity in introductory physics (NoNIP), analogous to the natures of negativity developed in the context of algebra. The intention is to provide a framework that can help researchers characterize and address the mathematical conceptualization of signed quantity in introductory physics. We conclude that the natures of negativity should be explicitly addressed in the context of introductory physics and calculus. We provide recommendations that can support the use of the NoNIP framework in the context of these courses.

A Model of the Natures of Negativity

The first generation of the natures of negativity for introductory physics was based on the natures of negativity described by Vlassis (Vlassis, 2004). We developed survey items to help map the algebra natures to a physics context—one survey question for each of the three natures in two contexts: mechanics quantities and E&M. The first survey item probes student understanding of the unary nature of the negative sign, the second probes the symmetrical nature, and the third, the binary nature (see Table 1). Table 2 presents all three mechanics items for reference.

We found that most uses of the negative sign typically found in introductory physics courses could be categorized using the map summarized by Vlassis. By using a mathematics-based sorting theme, however, we found we often lost the nuances of the physics described by the math; for example, we found that both scalars and vectors might be placed in the same broad category, despite the importance in physics of distinguishing between vector and scalar quantities. Because our intent was to encode both physical and mathematical meanings of the negative sign, we started from scratch keeping the physics as our primary guide.

Because of the importance of the difference between scalar and vector quantities in physics, our first attempt at mapping the natures of negativity in introductory physics began with a broad categorization based on whether quantities were vector or scalar. Some vector relationships are exclusively opposite in nature, such as Newton's Third-Law pairs, and the relationship between force and potential, $\vec{F} = -\vec{\nabla}U$. It was determined that this was the only possible categorization for 'complete' vector quantities, rather than vector components; in this case, 'opposite' indicates

Table 2: Questions representing different algebraic natures of negativity in introductory mechanics

Unary *structural signifier*	Symmetrical *operational signifier*	Binary *operational signifier*
Direction of a vector component	**Signifies work results in decreasing the system, energy, not increasing it**	**Position relative to an origin**
An object moves along the x-axis, and the acceleration is measured to be $a_x = -8$ m/s^2. Describe in your own words the meaning of the negative sign in the mathematical statement "$a_x = -8$ m/s^2".	A hand exerts a horizontal force on a block as the block moves on a frictionless horizontal surface. For a particular interval of the motion, the work W done by the hand is $W = -2.7$ J. Describe in your own words the meaning of the negative sign in the mathematical statement "$W = -2.7$ J".	A cart is moving along the x-axis. At a specific instant, the cart is at position $x = -7$ m. Describe in your own words the meaning of the negative sign in the mathematical statement "$x = -7$ m".

that the vectors in the relationship point in opposite directions (i.e., they are anti-parallel). Another vector-related category was for vector component quantities, and had two sub-categories: quantities for which the negative sign indicates the direction of the component relative to a coordinate system (such as v_x, F_x, E_x, or Δp_x), and one-dimensional relationships similar to the 'opposite' category for vector quantities described above.

Scalar quantities were subdivided into four categories: a) Amount; b) Opposite/opposing; c) Difference/change; and d) Label. The subcategory Amount is reserved for quantities for which we consider a negative amount of a thing. Such quantities are rare, and are only derived (not base) quantities. Total and potential energy, as well as scalar product quantities such as work and electric or magnetic flux were categorized in this way. Scalars in the opposite/opposing category include charge (as positive and negative charge are opposite types of charge) and relationships such as Faraday's Law, and $\Delta V = W/q$. The Difference/change category was used for time rates of change of scalar quantities, where the sign of the quantity indicates an increase or decrease, and for changes in a system such as energy change, ΔE, or temperature change, ΔT. Finally, the Label category was used only for charge; the sign of a charge tells us the type of charge, while the charge of an object tells us the type of charge in excess on the object.

Although this categorization did allow for the differentiation of vector and scalar quantities, we found it unsatisfactory overall. There seemed to be more variation within categories than between them, and we found that it placed quantities with similar physical characteristics (such as relationships that fell into the "opposite" categories for both scalars and vectors) into different categories. We also found that this categorization scheme did not allow for differentiation between uses of the negative sign as an operator. Moreover, quantities for which the negative sign has multiple interpretations (e.g., mechanical work as a scalar product and as measure of system energy change) were poorly represented by this categorization. Because our focus was on physics quantities rather than relationships between quantities, it was difficult to categorize models for which a negative sign is not an explicit part of the relationship. Finally, we recognized that there were quantities such as the product $f(x)dx$ that were not well-represented in this scheme. Physics and mathematics education researcher had indicated that products of integrands and differentials pose challenges for students when one or both of the factors are negative (Bajracharya, Wemyss, & Thompson, 2012; Sealey & Thompson, 2016).

The first two authors employed a modified card-sorting task for a second attempt at creating an expert version of the natures of negativity in physics, in which we again brainstormed and sorted physics quantities *and* relationships typically introduced in introductory physics. Categories were created based on the overarching similarities without first dividing quantities based on whether they were vector or scalar in nature. We created several sub-categories for each main category, largely to account for nuances in physical meaning. We determined three basic categories: *Direction (D), Opposition (O),* and *Change (Ch)*. A fourth category, *Compound (Co)* was added for instances when multiple meanings are assigned to the negative sign in a single expression or concept. Table 3 shows the results of this effort to create a map of the natures of negativity in introductory physics. We have surveyed introductory physics textbooks, checking that described signed quantities can be categorized satisfactorily with our scheme. We conducted expert interviews with physics instructors to ensure that this map of natures of negativity is valid for describing a majority of signed quantities in introductory physics and proposes a categorization that makes sense in the introductory physics context. A number of small changes were made based on these interviews, resulting in the form included in this paper. Additionally, we conducted expert interviews with mathematics faculty who were familiar with the physics contexts, including one math education researcher, to ensure that mathematical validity of this categorization; a repeating theme from these interviews with math experts was the importance of the meaning of 'zero' or 'origin' in each of these cases. This also indicated to us that reasoning about the sign of every quantity (not just reasoning about negativity) was important for more complete understanding of physics quantities.

We note that the *Direction* and *Opposition* categories are supported by the categories isolated by mathematics education researcher Chiu. In their study, they identified three categories of metaphorical reasoning that both middle school students and undergraduate and graduate mathematics and engineering majors used during problem-solving interviews—motion, manipulation of objects/opposing objects, and social transaction (associated with the experiences of giving and exchanging) (Chiu, 2001). While these are metaphors in mathematics, they are in fact *contexts* in physics in which a conceptual mathematical understanding is essential for learning the physics. The entire content of mechanics is focused on actual motion in space (not motion along a number line). Phenomena that arise due to the parallel or antiparallel orientations of two quantities are ubiquitous throughout physics (i.e. speeding up/slowing down, friction and air resistance, electromagnetic induction). Direction and Opposition are central natures of signed quantities in physics.

The *Direction* category is used largely for components of vector quantities. We differentiate between **1. Location** (for which the sign tells us the position relative to an origin), **2. Direction of motion** (typically used for a vector component, where sign indicates direction of motion relative to a coordinate system), and **3. Other vector quantities** (where the sign of a vector component tells us the direction of that component relative to a coordinate system, but when motion is not an intrinsic quality of the vector quantity). We consider subcategories **2** and **3** separately, as direction of motion is readily apparent and observable. Finally, we consider **4. Above/below reference** for *scalar* quantities such as electric potential and temperature, for which the zero of the quantity is an arbitrary reference point.

For the category *Opposition*, we consider quantities for which a negative sign implies opposite direction or relationship. **1. Opposite type**, as positive and negative charge are "opposite" types

Table 3: The Natures of Negativity in Introductory Physics, a map of the different uses of the negative sign in introductory physics

(D) Direction	(O) Opposition	(Ch) Change	(Co) Compound
1. Location	**1. Opposite type**	**1. Removal (operator)**	**1. Scalar rates of change**
x	*Q (charge)*	$0-(-5\mu C)$	$\frac{d\Phi}{dt}$
2. Direction of motion	**2. Opposes**	**2. Difference (operator)**	**2. Base + change**
$v_x, \Delta x$	$\vec{F}_{12} = -\vec{F}_{21}$	$E_f - E_i$	$\phi + \frac{d\phi}{dt}t$
p_x	$\vec{F} = -\vec{\nabla}U$	$\vec{p}_f - \vec{p}_i$	$\vec{v} + \vec{a}t$
3. Other vec. quant. comp.	$\mathscr{E} = -\frac{d\Phi_B}{dt}$	**3. System scalar quantities**	**3. Products** $f(x)dx$
E_x, B_x	$\vec{F} = -k\vec{r}$	$\Delta K, \Delta E$	$E(r)dr$
F_x, L_z	**3. Scalar products**	ΔS	$P(V)dV$
a_x	$W = \vec{F} \cdot \Delta\vec{x}$	**4. Scalar, vector change**	**4. Models**
$\Delta p_x, \Delta v_x$	$\Phi = \vec{B} \cdot \vec{A}$	$\Delta E = E_f - E_i, \Delta V = V_f - V_i$	$W_{net,ext} = \Delta E$
4. Above/below reference		$\vec{\Delta p} = \vec{p}_f - \vec{p}_i$	$\vec{F}_{net} = m\vec{a}$
T (temperature)			$\Delta U = Q - W$
V (electric potential)			

of charge, and obey the mathematical relationship of $+q+(-q)=0$ (i.e., adding equal amounts of opposite types of charge leads to a system with no net charge). For the the subcategory **2. Opposes**, we consider scalar and vector relationships between quantities that indicate that the quantities oppose each other in direction or change, such as members of a Newton's Third Law force-pair.

The category *Change* encompasses both the meaning of the sign of the change of a quantity as well as the negative sign as an operator that signifies a change in a quantity. (**1. Removal (operator)**). We may also use the negative sign to signify that we are taking a difference between two quantities (as in determining the change of a quantity), as described by **2. Difference (operator)**. Subcategory **3. System scalar quantities** describes quantities that characterize change in a system, such as changes in energy or entropy. For **4. Scalar, Vector change**, when students are asked to calculate a change in a quantity such as energy or momentum, they must first account for the signs of the initial and final quantities, then successfully subtract one from the other and make sense of the result.

Finally, the *Compound* category covers instances when the negative sign spans more than one category, or that require one to 'keep track' of several signs when making sense of a quantity or relationship. **1. Scalar rates of change**, **2. Base + change** (base quantities that are increased or decreased by the addition of a change; the concept of accumulated change is ubiquitous in physics), and **3. Products** $f(x)dx$ (products of integrands and differentials). We also include in this category **4. Models**, to account for models that require sensemaking of a negative sign. The Work-Energy Theorem, where the sign of $W_{net,ext}$ indicates whether a system gains or loses mechanical energy, is an example of such a model.

Applying the framework

In this section, we use the NoNIP as an analytical lens through which to view recently published studies in physics and calculus and that mostly involve advanced physics or math students.

Bajracharya, Wemyss, and Thompson (2012) investigated upper-division student understanding of integration in the context of definite integrals commonly found in introductory physics, but with physics context stripped from the representation: the variables typically used in physics contexts were replaced with x and $f(x)$ (Bajracharya et al., 2012). Their results suggest difficulties with the criteria that determine the sign of a definite integral. Students struggle with the concept of a negative area-under-the-curve, and in particular negative directions of single-variable integration. Sealey and Thompson (2016) interviewed math majors to uncover how they made sense of a negative definite integral. Undergraduate (beyond introductory) and graduate mathematics students had difficulty to make meaning of a negative differential in the context of integration (Sealey & Thompson, 2016). The struggles these researchers described can be seen through the lens of NoNIP as struggle with the product of the integrand, $f(x)$, and the differential, dx (Co.4 in NoNIP). The negativity of the integrand (D in NoNIP) was less of a struggle for the students in these studies than was the notion of a negative differential (Ch in NoNIP), which has application throughout physics.

Hayes and Wittmann (2010) report on an investigation in a junior-level mechanics course of negative signs and quantities associated with the equation of motion of an object thrown downward, with non-negligible air resistance (Hayes & Wittmann, 2010). The equation of motion is $ma = mg - bv$, or $m\frac{d^2x}{dt^2} = mg - b\frac{dx}{dt}$, where the initial velocity exceeds the terminal velocity so the object is thrown downward and slows down—the velocity and the acceleration oppose each other initially. The student interviewed struggles with treating one-dimensional acceleration as a signed quantity, and feels there should be an additional negative sign included to indicate that the acceleration is opposing the motion. The authors conclude that the multiple natures of the negative sign are a source of cognitive conflict that the student can't resolve. Mathematics education researchers have found that younger students tend to assign only natural numbers to literal symbols or to treat expressions such as $-x$ as if they represent solely negative quantities (Christou & Vosniadou, 2012; Lamb et al., 2012). Although the students in the Hayes and Wittman study are well beyond Calculus II, it appears they revert to a more primitive treatment of vector quantities when they encounter a challenging context that calls on multiple meanings of the negative sign. Seen through the lens of NoNIP, minus is an operator, and negative signs are used to represent many mathematical objects in physics. In this context the student struggles with D.3 and D.2 in the contexts of one-dimensional acceleration and velocity. The negative sign that modifies the bv term is used as O.2, to indicate that the force is in the opposite direction to the velocity. Combining terms, the students struggle to make sense of the equation of motion. The cognitive load associated with the individual terms contribute to a higher-level struggle of making physical sense (Co.5).

In their study of negativity in junior level Electricity and Magnetism, Huynh and Sayre (2018) describe the in-the-moment thinking of a student solving for the direction only of a positive and negative charge distributed along a line symmetrically about the origin (Huynh & Sayre, 2018). The solution involves an algebraic superposition of the field due to each charge individually. In their study the authors focus on the student reasoning about the sign of the the electric field vector component along the axis of symmetry in three regions of space—to the left of one charge, between the two charges and to the right of the other charge. The authors detail the students' development of an increasingly blended approach that is situated in a mental space informed by both mathematical and physical concepts. The student starts reasoning about the direction of the field by (unintentionally) combining multiple natures of negativity into one, using the canceling procedure that two negatives make a positive, without considering the source of each negative sign. In Coulomb's law, signs come in associated with the charges, the unit vector and the electric field vector direction. Collapsing the signs using arithmetic rules is a common approach first tried by the students in this study, which focuses on the multiplicative rules of signed numbers rather than the physics of the meaning of the signs. Next the student rarefies his approach as he considers more carefully the natures of negativity in the context of the problem. Seen through the lens of NoNIP we can see evidence of the student first conflating the natures superficially; the authors describe, "...he decides to absorb the destructive meaning...into the opposite meaning...and changes the second negative sign to a plus sign...however he didn't consider the...relative direction...leading to...the opposite sign of the correct answer." Then as he slows his thinking, first recognizing D.2 and D.3, the unit and electric field vectors and as sources of negative signs, the student says, "...I should have figured it out...which direction it is. This is exactly what is changing signs." After reconciling the basic level, then he struggles with O.2, the authors describe that the student "has successfully affiliated the sign's meaning to the relative direction...electric fields and x-hat." The authors conclude, and we agree, that the most sophisticated challenge occurs when these natures are combined in which three natures of the negative sign must be made sense of in the context of a single equation, Co.5. This example illustrates the challenges associated with reasoning about the natures of negativity even for strong majors, and reveals a hierarchy that lends plausibility the NoNIP model being representative of emergent expert-like reasoning.

Implications for instruction

Student difficulties are embedded in natures of negativity that can be, and we argue should be, explicitly addressed in the context of introductory physics and calculus. We suggest that instructors familiarize themselves with the many jobs that the negative sign does in introductory physics courses, and help students recognize the varied natures of signed quantities. The NoNIP framework can help. We offer two suggestions as a start:

1. In problems associated with motion, aligning the positive coordinate axis with the direction of motion eliminates the need for signed quantities when discussing velocity. This choice, however, could be a missed opportunity to distinguish between orientation and sense. The opposite coordinate choice can prime students to consider the signed nature of position, velocity, and subsequent vectors quantities they encounter.

2. Applications that involve quantities that are inherently signed quantities should be prefaced with a negative sign when the quantity is negative, and a positive sign when positive. Priming students in a math course to expect that real-world quantities have signs that carry meaning, and that 'no sign' is a different kind of quantity than a positively-signed quantity, will help better prepare students. These quantities in physics include, but aren't limited to: position, displacement, velocity, acceleration, force, and work.

In addition to enriching subsequent physics learning, a focus on natures of negativity in physics contexts can also enrich the corequisite mathematics learning. Sealey and Thompson report on a context in which physics helps math students make sense of negativity in calculus. The researchers observed that invoking a physics example of a stretched spring helped catalyze sense making—the physics helped them to make sense of an abstract binary nature of the negative sign (Sealey & Thompson, 2016). We suggest that there is a symbiotic cognition possible in which both mathematics and physics learning can be enriched by conceptualization of the other. We present NoNIP as a representation of signed quantity providing a step in that direction.

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