What Are You Looking At? Shape Thinking and Eye-Tracking

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Previous research has illuminated and defined meanings and understandings that students demonstrate when reasoning about graphical images. This study used verbal and physical cues to classify students' reasoning as either static or emergent thinking. Eye-tracking software provided further insight into precisely what students were attending to when reasoning about these graphical images. Eye-tracking results, such as eye movements, switches between depictions of relevant quantities, and total time spent on attending to attributes of the graph depicting quantities, were used to uncover patterns that emerged within groups of students that exhibited similar in-the-moment meanings and understandings. Results indicate that eye-tracking data supports previously defined verbal and physical indicators of students' ways of reasoning, and can document a change in attention for participants whose ways of reasoning over the course of a task change.

Key Words: Static Shape and Emergent Shape Thinking, Quantity, Covariation, Eye-Tracking

Many students entering calculus have been indoctrinated into a rule-based mathematics that uses rote memorization, but this can lead to struggles when students face problems that contain dynamic phenomena. To move students away from merely using memorized methods, they must be provided with tasks which require them to reason about individual quantities and how two quantities' magnitudes vary over time (Moore & Thompson, 2015; Stevens & Moore, 2017; Thompson, 2011).

In the past, researchers have been restricted to categorizing students' meanings and understanding of concepts based on verbal and physical cues. Recently, researchers have started using eye-tracking technology, which allows the addition of visual cues by tracking student fixations while they reason about tasks (i.e. Alcock, Hodds, Roy, & Inglis, 2015). Although eye-tracking studies have been conducted in undergraduate mathematics education, those studies have focused on the use of static images, such as describing how experts and novices read proofs (i.e. Alcock et al., 2015).

There has been much research on how students reason quantitatively and covariationally (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Monk, 1992; Moore & Carlson, 2012; Moore & Thompson, 2015). This research aimed to delve deeper into past results by re-creating quantitative and covariational graphical interview tasks synced with eye-tracking. The research questions addressed in our study are:

1. How do students fixate on various graphical attributes depicting quantities relevant to the corresponding task?
2. Is there a relationship between students' fixation patterns and observed meanings evidencing static or emergent shape thinking?

Background

Thompson, Hatfield, Yoon, Joshua, & Byerley (2017) presented statistical data about U.S. high school performance when asked to create a trace of a graph while watching an animation depicting covarying magnitudes (see Figure 1c). Only 23% of U.S. high school teachers were able to at least create a semi-accurate trace of the graph. They reported a high correlation ($p < .0001$) between the creation of the correct initial point and providing an accurate graph. Thompson (2017) noted that a potential limitation was that teachers could not simultaneously look at the animation and their paper on which they were sketching their graph.

Thompson et al. (2017) revealed problems, but insight into student reasoning about graphing was minimal. Moore and Thompson (2015) leveraged Piaget's notions of figurative and operative thought together with quantitative and covariational reasoning to better describe how students reason about graphs. Static shape thinking, according to Moore and Thompson (2015), is defined as seeing a graph "as an object in and of itself, essentially treating a graph as a piece of wire (graph-as-wire)" (p. 784). If a student interprets a graphical representation as "graph-as-wire," they see the wire as an entire unit with no individual components (multiplicative objects) making up the wire. Equations, function names and rules are "facts of shape" (p. 785). It is important to note that static shape thinking often suffices to evaluate procedural type problems. For example, memorizing shapes and rules, such as the first few terms of a Taylor series, can be a productive way to avoid re-inventing the wheel each time a new problem is presented (Martin & Thomas, 2017). However, static shape thinking becomes a problem when it inhibits a student's ability to reason about and conceive of the various aspects involved in dynamic graphical images.

In contrast to static shape thinking, emergent shape thinking "involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (Moore and Thompson, 2015, p. 785). This mode of thinking is rooted in students' abilities to reason in terms of quantities (quantitative reasoning) and how those quantities vary in tandem (covariational reasoning). By *quantity,* we are referring to a cognitive construct of a measurable attribute of an object or phenomenon (Thompson, 1994; Thompson, 2011). It is important to note that this type of reasoning is not an inherent feature of a situation; just because a student is immersed in a dynamic task that may seem to beg for covariational reasoning does *not* mean that the student will *conceive* of the situation in terms of covarying quantities.

Methods

Overall Study Design

Eleven student volunteers were asked to participate in two task-based, semi-structured, clinical interviews (Goldin, 2000) lasting no longer than two total hours. Since the prior mathematical knowledge of participants varied, the total amount of time to complete all tasks varied. Anticipated course grades were no lower than a C average for any participant, and students not recommended by their instructors based on inability to communicate were also not contacted. In total, eleven students completed twelve tasks presented on a computer monitor. For the purpose of this report we focus on three tasks presented in Figures 1a, 1b and 1c.

An over-the-shoulder camera captured students' note-taking and gestures. Tobii eye-tracking software (Tobii, 2018), collected eye fixation data. Audio from these sources was used to sync the camera and eye-tracking videos. Key moments were transcribed, including verbal utterances and relevant gestures. Raw data from the eye-tracking software consisted of coordinate points indicating participants' visual attention to specific locations on the monitor associated with a timestamp.

Areas of Interest (AOIs)

Areas of Interest (AOIs) were constructed prior to the interviews (see individual task protocol below for tasks and their corresponding AOIs). AOIs were not visible to the participants. Many AOIs were created to include single attributes depicted on the graph that students could conceive as relevant quantities. Task 1, for example, prompted students to identify *x* segment associated with a point, so the AOIs covered *x* attributes, *y* attributes, the point itself, and so on (see Figure 1d). Defining AOIs allowed researchers to collect information regarding eye movement, such as switches. A switch between AOIs is counted each time a student's fixation moves from one AOI to another provided that the student fixated within the second AOI within 0.5 seconds of their fixation leaving the first AOI. If the student fixated within one AOI, fixated within a second AOI in less than 0.5 seconds after leaving the first AOI, and then fixated within a third AOI in less than 0.5 seconds after leaving the second AOI, then that was counted as two switches, one switch from the first to second AOIs and another from the second to third AOIs.

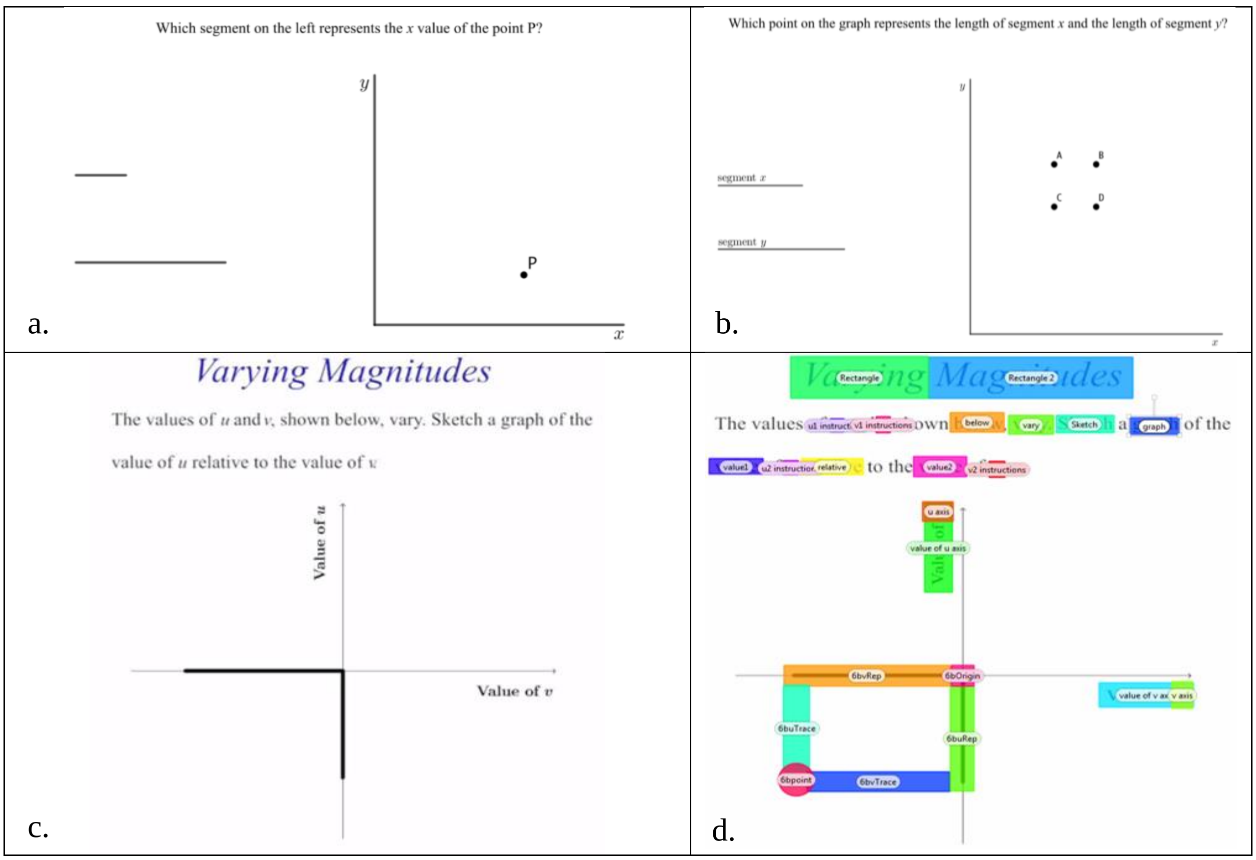

Figure 1. Screenshots of Tasks 1, 2, 6, and AOIs defined for Task 6.

Analysis and Coding

In Tasks 1 and 2 (Figure 1a and 1b, respectively), students were marked as correct if they indicated the bottom segment as corresponding to the *x*-value of point P, and indicated point C as representative of the two segments, respectively. Results for Task 6 were only coded as correct (see Figure 1c) if the students created a graph that closely resembled the correct trace (correct number of maximums, minimums, correct placement of initial point).

Potential indicators for quantitative reasoning included verbal and physical cues. Verbal cues for quantitative reasoning included words such as “length” or “distance.” Gesturing with hands, such as spreading out thumb and index finger over a depicted segment, was also coded as a potential indicator of quantitative reasoning. Although it was technically possible that such a gesture might merely indicate visually transferring a line segment without explicit reference to measurement, such gestures were frequently paired with the participant acknowledging the “length” of the segment.

Indicators for quantitative reasoning coincide with emergent shape thinking. The lack of these indicators can be indicative of static shape thinking. In addition, the data was also coded for students’ use of named shapes, such as “quadratic” as meaning graphs that increase and decrease. Finally, these potential indicators for static and emergent shape thinking were compared to eye-tracking fixation data (pulled from AOIs shown in Figure 1d) to determine if potential patterns emerged within different ways of thinking.

Results

Table 1 shows general results for each participant. Of the six participants who correctly responded to both Tasks 1 and 2, only P08 was unable to create an accurate trace in Task 6. This data supports Thompson’s (2017) conclusions that a participant who correctly plots the initial point for Task 6, indicated by a participant’s ability to correctly interpret and create a point in Tasks 1 and 2, respectively, is more likely to create an accurate graph.

Table 1
Overall Results for Correctness and Indicators of Potential Quantitative Reasoning (QR)

Participant	Task 1 Correct	QR	Task 2 Correct	QR	Task 6 Correct
P01	✓				
P02	✓		✓	✓	✓
P03			✓	✓	✓
P04	✓	✓		✓	✓
P05	✓			✓	
P06	✓	✓		✓	
P07	✓		✓	✓	✓
P08	✓	✓	✓	✓	
P09	✓	✓	✓	✓	✓
P10	✓	✓	✓	✓	✓
P11	✓	✓	✓	✓	✓

Neglecting Depictions of Quantities

P03 was the only participant who answered incorrectly on Task 1 but answered correctly on both Tasks 2 and 6. Eye-tracking data (Figure 2) offers a possible explanation. P03 incorrectly interpreted the task, which led to the choice of the top segment and the *y*

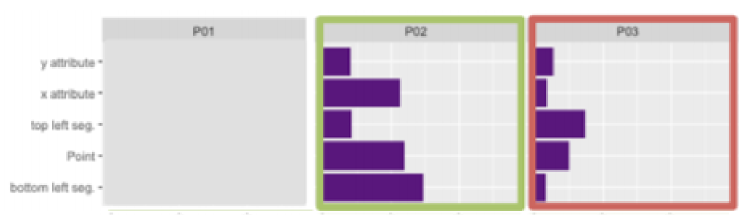

Figure 2. Total participant time on selected AOIs in seconds for Task 1.

attribute of the graph. As expected, Figure 2 indicates a lack of attention by P03 to attributes of the graph corresponding to the *x*-component of the point when compared to attributes of the *y*-component of the point. It is also apparent that almost all the remaining participants, similar to P02 shown, attended to the *x* attributes for a larger amount of time.

Figure 3 gives the number of switches between AOIs. It is evident that few switches from one *x* attribute of the graph to another occurred for P03. Figure 3 shows exactly this - very few switches other than from the top left segment to the point. In comparison, P02 indicates a satisfactory number of switches between relevant *x* quantities.

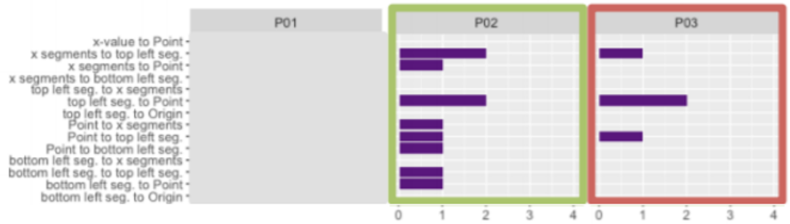

Figure 3. Task 1 switch count.

Changing Fixation Patterns

Eye-Tracking data also shows variance in participant's attention to graphical in real-time as their ways of reasoning about an image changed. For Task 2, we present an example of a fixation timeline produced by P07.

Figure 4. Fixation timeline for P07 during Task 2.

For Task 2, P07 had a large gap in his fixation timeline, with drastic difference in the color schemes before and after. The gap for P07 indicates that he was not fixated within any defined AOI. Video data indicated that he was attending to the interviewer while she added the word *relationship* to the task ("Which segment on the graph represents the *relationship* between the length of segment *x* and the length of segment *y*"). The color scheme on the left side shows P07 looking back and forth between segments *x* and y, then back and forth between x attributes (horizontal aspects of the graph) and y attributes (vertical aspects of the graph). The change in color scheme for the latter half of the fixation timeline shows attention to relevant aspects of the graph in a more meaningful order (segment *x* to *x* attributes and segment *y* to *y* attributes). His initial lack of awareness of coordination of quantities was resolved by the insertion of the word relationship in the task instructions. A look at P07's dialogue during the gap confirms this change in reasoning:

> I: Which point on the graph represents the relationship between the length of segment *x* and the length of segment *y*.
> P07: I'd say C because if you take the *y* segment and you match it up right here [right index at C, right thumb on *x* axis below C] it would be about that length and since the *x* is shorter it would probably be about at C [right index finger at C, right thumb on the projection of C onto the *y* axis].

When considering Task 6, P01 was incorrect, as shown by the red border in Figure 5. P02 was correct (green border in Figure 5). P03's total time spent on AOIs (yellow border in Figure 5) look very similar to P01's total time spent on AOIs. Yet, P03 was correct in his response to Task 6.

Unlike the total participant time, the switch count for Task 6 (Figure 6) provides two *very* different results for P01 and P03. Although the two participants spent a similar total amount of time on the *x* and *y* representations, we see from the switch count that P03 was actively switching between the AOIs (12 times) while P01 only made *one* switch between representations.

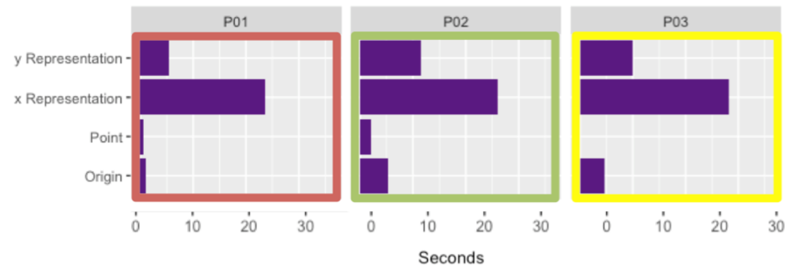

Figure 5. Total time spent on relevant AOIs for Task 6.

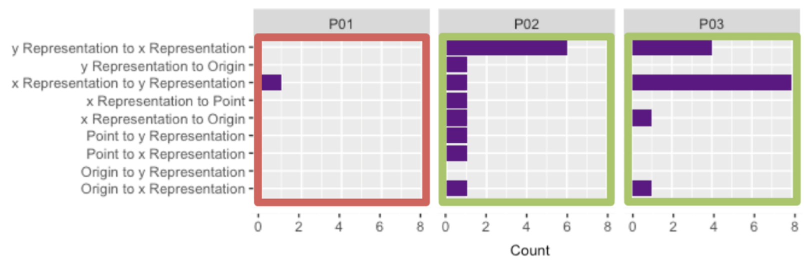

Figure 6. Switch count between relevant AOIs for Task 6.

Transition from Static Shape Thinking to Emergent Shape Thinking

During Task 6, P02's verbal and physical cues combined with his eye-tracking data yielded results evident of a possible transition from static to more emergent ways of reasoning.

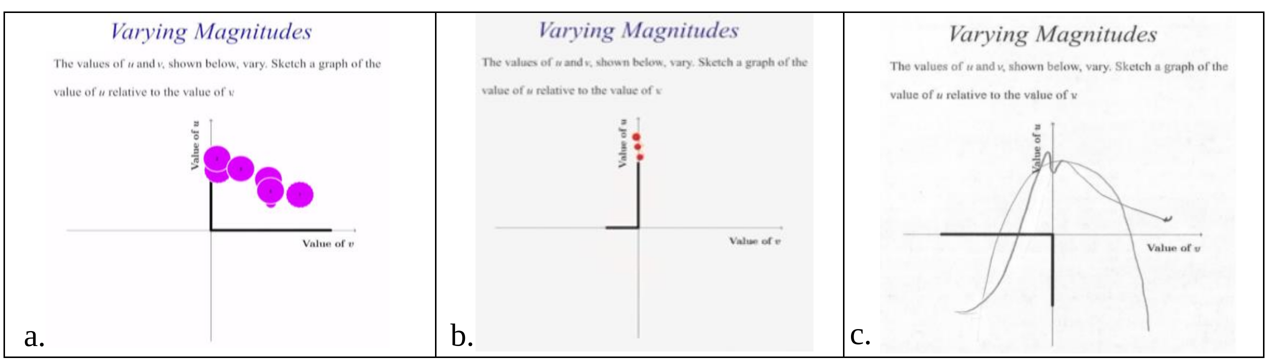

Figure 7. Screenshots of P02's fixation patterns, a. and b., and attempted Graphs in c.

P02's eye-tracking indicated that he was at moments following the moving point location resulting from the coordination of at least the endpoints of *u* and *v* line segments (see Figure 7a). Yet P02's dialogue indicated that the reasoning upon which he based his initial graph was more static in nature. "I think what it is is they… It's about like this [drawing parabola in Figure 7c] if we were to continue on as it would go on. I think it's just an upside down parabola, so *y* equals negative *x* squared is what I think…" He then drew the concave down parabola in Figure 5c.

A few moments later, however, P02 begin to follow the moving point location on the screen with his pencil. While P02 was still trying to attend to the perceived point created by *u* and *v*, he attended more so to the vertical attribute of *u* than before (Figure 7b). After making multiple up and down movements with the pencil on screen, the participant decide that his parabola was insufficient, and draw the more accurate image in Figure 7c over it.

Discussion

Conclusions

Participants' fixation counts alone were not necessarily indicative of whether they correctly interpreted an image, nor were they indicative of the ways of reasoning in which they were engaged. When paired with switch counts, as was the case when comparing P02 and P03, an ability to switch fixations between graphical attributes depicting quantities relevant to Task 1 appears to be related to the participants' ability to reason correctly about the task. Participants who correctly answered a given task generally had a higher volume of switch recordings, indicating a greater attention to the relationship between quantities represented in the task.

P07's timeline for Task 2 demonstrates an instance where the individual entered a state of disequilibrium through verbal cues that caused a change in fixation to relevant quantities and the relationships between them. Over the course of Task 6's animation, P02 engaged in static shape thinking to initially conceive of the shape of the graph. Even though he had attended to some variation, as demonstrated by the eye-tracking, he proceeded to assign a specific function to the graph, a parabola in this case. We anticipated that students who indicated emergent shape thinking might fixate on a moving point location resulting from the coordination of varying the values of the lengths of u and v. Yet, P02's shift to emergent shape thinking during Task 6, resulted after his fixations had transitioned primarily the value of u. When attending to the variation in u he was able to hold in mind the variation of v and eventually produce a more accurate trace of the graph. This demonstrates that students need not continually switch back and forth between varying depictions of quantities to successfully engage in emergent shape thinking.

However, the low switch count for P03 in Task 6 (Figure 6) shows that although the participant was attending to relevant quantities for an extended period of time (Figure 5), he was not actively moving his attention between the quantities. P02 did indeed switch (see Figure 6) and apparently engaged in enough relevant switching for him to produce an accurate graph with reduced switching while graphing. But, a lack of switch counts for students may be indicative of a lack of attention to the coordination of quantities.

Future Work

Eye-tracking software is a new tool that is emerging in mathematics education literature, which leaves a wide range of possibilities for further research on the aspects discussed in this study. One limitation of this study is that results need not generalize to other students, and therefore, a larger sample size is needed.

Eye-tracking results can also be used to develop instructional videos or tasks that better equip students to reason in terms of quantities and dynamic situations. Currently, a research team is working on an NSF funded project that is using eye-tracking to investigate how students are attending to the videos (see acknowledgment; calcvids.org).

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References

Alcock, L., Hodds, M., Roy, S., & Inglis, M. (2015). Investigating and improving undergraduate proof comprehension. *Notices of the AMS, 62,* p. 742-752.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS issues in mathematics education: Research in collegiate mathematics education III* (Vol. 7, pp. 114-162).

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic vents: A framework and a study. *Journal for Research in Mathematics Education*, 33, 352-378.

Goldin, G. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. Lesh (Eds.), Handbook of research design in mathematics and science education (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Martin, J. & Thomas, M. (2017). *Effects of interactive virtual number lines in Taylor series graphs to support pointwise convergence for calculus students.* Manuscript submitted for application.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes* (Vol. 25, pp. 175-193). Washington DC: Mathematical Association of America.

Moore, K., & Carlson, M. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior, 31,* 48-59.

Moore, K., & Thompson, P. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education.* Pittsburgh, PA: RUME.

Stevens, I., & Moore, K. (2017). A case study: When graphs contain everything. *Abstracts of Papers Presented to American Mathematical Society.*

Thompson, P. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26,* 229-274.

Thompson, P. (2011). Quantitative reasoning and mathematical modeling. In L.Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education, WISDOMe Monographs* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P., Carlson, M., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. Steffe, K. Moore, L. Hatfield & S. Belbase (Eds.), Epistemic algebraic students: Emerging models of students' algebraic knowing (pp. 1-24). Laramie, WY: University of Wyoming.

Thompson, P., Hatfield, N., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior, 48,* 95-111.

Tobii Pro. (2018, August 1). Retrieved from http://www.tobii.com/.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics, 34,* 97- 129.