Variational Reasoning Used by Students While Discussing Differential Equations

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In this study we investigated how a small sample of students used variational reasoning while discussing ordinary differential equations. We found that students had flexibility in thinking of rate as an object, while simultaneously unpacking it in the same reasoning instance. We also saw that many elements of covariational reasoning and multivariational reasoning already discussed in the literature were used by the students. However, and importantly, new aspects of variational reasoning were identified in this study, including: (a) a type of variational reasoning not yet reported in the literature that we call "feedback variation" and (b) new types of objects, different from numeric-quantities, that the students covaried.

Keywords: Differential equations, covariation, multivariation, rate, feedback variation

Ordinary differential equations (DEs) are complex constructs that require reasoning about an interconnected set of relationships. A few researchers have provided deep insight into how students broadly understand DEs (e.g., Habre, 2000; Keene, 2007; Rasmussen, 2001). The literature on DEs reveals the importance of two mathematical concepts: function and rate of change. The importance of *function* is evident in results concerning student understanding of: solutions to DE's (e.g., Arslan, 2010; Dana-Picard & Kidron, 2008; Rasmussen, 2001; Raychaudhuri, 2014), understanding the quantities involved in DE's (Stephan & Rasmussen, 2002, Raychaudhuri, 2008), and existence and uniqueness theorems (Raychaudhuri, 2007). Donovan (2007) noted that when students were able to conceptualize first-order DEs as functions, they were afforded rich ways of reasoning about the solutions. Student notions of *rate* have emerged in the literature in a few different ways. For instance, Keene (2007) identified reasoning about time as a dynamic quantity in relation to other quantities as an important way of reasoning about solutions. Rasmussen & Blumenfeld (2007) outlined how students could use their notion of rate of change to construct solutions to systems of differential equations. In addition, Whitehead and Rasmussen (2003) identified *rate use*, where students used rate as a tool for determining solution functions.

These studies have provided important insight into how students reason about DEs and their solutions. However, there is a key component implicit to much of this work that has not been directly studied: how students use *variational reasoning* while thinking about DEs. Knowing how students use such reasoning could give insight to instructors both in terms of recognizing and eliciting student reasoning. In response to this gap in research, the study we present in this paper was guided by the question, "How do students use variational reasoning when interpreting and discussing DEs?"

Variational Reasoning

We use the generic term "variational reasoning" to mean reasoning about *any* situation involving a changing quantity. The term "quantity" refers to an object with an attribute that can be measured (Thompson & Carlson, 2017, p. 425). We use *co*-variation to mean how *two* quantities change in relation to each other, based on the extensive focus in the literature on two-quantity covariation (e.g., Carlson et al, 2002; Confrey & Smith, 1995; Johnson, 2015; Moore et al, 2013; Saldanha & Thompson, 1998). Thompson and Carlson (2017) have recently published a new covariational framework based on previous covariation research, consisting of six reasoning levels. We use this framework, except for the first level, in which a student does not engage in any coordination. Instead, we retain the first mental action, *recognize dependence*, from Carlson et al.'s (2002) original framework. Consequently, our first level is *recognize dependence*, in which a student perceives two quantities as being dependent in some way. The second level (now from the new framework), *precoordination*, involves imagining two quantities changing, but "asynchronously" (p. 441), meaning that the person envisions a change in one quantity first, then a change in the other. The third level, *gross coordination*, contains an image of two quantities changing together, but in a generic way, such as "this quantity increases while that quantity decreases" (p. 441). The fourth level, *coordination*, involves "coordinat[ing] the values of one variable (x) with values of another variable (y)" (p. 441). The fifth level, *chunky continuous reasoning*, involves imagining continuous change, but always by completed intervals, or "chunks," of a fixed size. In the sixth level, *smooth continuous covariation*, the person envisions the changes in the two quantities "as happening simultaneously" and with "both variables varying smoothly and continuously" (p. 441). Note that we use this framework as a "descriptor of a class of behaviors" (p. 441), rather than as a judgment of overall ability. That is, a student's usage of one level does not imply the inability to use a higher level.

Next, we use *multi*-variation for situations in which *more than two* quantities change in relation to each other (Jones, 2018). For our purposes, we use three types of multivariation. *Independent multivariation* involves two (or more) independent quantities influencing a third quantity, but where the independent quantities do not directly influence each other. *Dependent multivariation* involves three or more interdependent quantities where a change in one typically induces changes in *all* other quantities in the system simultaneously. *Nested multivariation* involves a chain of related dependencies, like the structure of function composition, $z = f(y(x))$. We use the term "multivariational reasoning" to mean the reasoning one does about the quantities involved in one of these types of multivariation.

Methods

As a preliminary step to our study, we conducted a conceptual analysis for ourselves on how one might interpret basic DEs of the forms $y' = f(y)$, $y' = f(t)$, and $y' = f(t, y)$. This confirmed to us that there would likely be many aspects to variational reasoning in interpreting these equations. Encouraged, we conducted our study using pre-existing data that came from a series of five task-based, semi-structured interviews done with eight students enrolled in a traditional ordinary differential equation course (not taught by the researcher). This data was collected as part of an earlier investigation by one of the authors in an effort to explore the connection between ideas involving function and rate of change in relation to student understanding of differential equations. For the purposes of this study we identified four tasks within these interviews that had DEs matching the three basic forms, including two that had symbolically written DEs and two that had visual graphs associated with DEs. The four tasks are shown in Figure 1. Note that tasks I2T4 and I4T2 were adapted from the IODE curriculum (Rasmussen, Keene, Dunmyre & Fortune, 2017). For the purposes of this conference report, we chose a small sample of three of the eight students to analyze, based on those that were most talkative and that articulated their thinking.

We analyzed the interview data at two levels: holistically and per instance of student reasoning. The second author went line by line through the transcript to identify each instance of student variational reasoning, done by noting any time a student talked about two or more quantities at the same time. For each identified instance of variational reasoning, a timestamp and the associated utterance were recorded, as well as the type of variation (co-, multi-, or other) and the quantities involved. For covariational reasoning, the instance was coded according to the Thompson and Carlson (2017) framework, with the inclusion of the *recognize dependence* level. For multivariational reasoning, the instance was classified in terms of the type of multivariation. The conceptual analysis helped us be sensitive to certain ways students might using variational reasoning. Independently, the first author took a more holistic approach, identifying the general steps and lines of reasoning the student utilized to complete each task. Within each line of reasoning, the researcher identified the main types of variation, the objects being varied, and key mental actions associated with the approach. For each interview task, the two authors then met and discussed the findings until a consensus was reached.

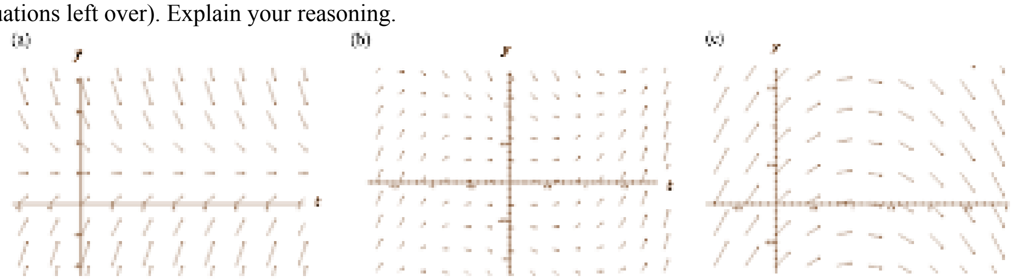
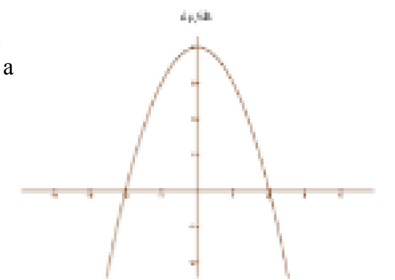
Interview 1 Task 2 (I1T2): What does the following differential equation mean to you? $P' = 3P$
Interview 1 Task 3 (I1T3): Suppose the equation $P' = 2P + 2t$ can be used to model the fish population in the campus duck pond. How might it be used to determine the number of fish in the pond at a given time t?
Interview 2 Task 4 (I2T4): Below are three different tangent vector fields and six rate of change equations. Without using technology, identify which differential equation is the best match for each tangent vector field (thus you will have three rate of change equations left over). Explain your reasoning.  $\frac{dy}{dt} = t-1$, $\frac{dy}{dt} = 1-y^2$, $\frac{dy}{dt} = y^2-t^2$, $\frac{dy}{dt} = 1-y$, $\frac{dy}{dt} = t^2-y^2$, $\frac{dy}{dt} = 1-t$
Interview 4 Task 2 (I4T2): Below you are provided with a graph of a rate of change equation rather than the equation itself (Note that *dy/dt* depends only on *y*). Figure out the long-term behavior of possible solution functions, illustrate your conclusions with a suitable graph or graphs, and state your conclusions about the long-term behavior of these solutions. 

Figure 1: The four interview tasks we analyzed for this paper

Results

We organize our results by first describing how the students in this study reasoned about rate, and how they generally employed covariational and multivariational reasoning. We then discuss novel findings regarding the variational reasoning used by the students, including (a) the importance of dependence, (b) a new type of variational reasoning, and (c) new types of objects used in covariation.

Student Interpretations of Rate

Our students often referred to variables such as P' and *dy/dt* by name, but also as a slope, derivative, value or ratio, and sometimes represented them graphically with direction vectors. Our purposes here were to understand what quantities students were varying and the variational reasoning they used to do so, and as long as the students seemed to see some equivalence between these different interpretations of derivative, or “rate,” we did not analyze according to which interpretation was being used. That is, while the students expressed these various images for rate, we did not focus on the specific properties of the representation (e.g., slopes, variables, vectors, etc), but rather on how the students reasoned about changes in the values associated with those objects. Categorically speaking, the students used their different understandings of rate in two key ways: as a single quantity in its own right and as a quantity that could be unpacked to indicate how the values of two distinct quantities changed in relation to one another. These two ways of using rate were often associated with different relationships in the DE’s.

Students in our study often used rate as a single quantity/object with a numeric value when reasoning about the relationship explicitly defined in the DE. For instance, while discussing $P' = 3P$, Student 1 noted that the DE "is indicating that as x goes towards infinity the rate of change is increasing dramatically, it gets bigger as time goes on." Similarly, on the same task Student 2 noted "as P increases, or as t or whatever it is related to increases, the rate of change increases." Both of these students used the DE to determine how the value of one variable was related to the value of the other. While they were both sensitive to P' being a rate of change (of P), their reasoning did not rely on this intrinsic characteristic of the DE; their reasoning was similar to thinking of the relationship between y and x in the equation $y = 3x$ (where y and x share no other inherent relationship).

On the other hand, the students also often unpacked rate in different ways. For task I1T2, Student 1 used P' to indicate the general behavior of the solution function: "as x increases P is going to increase faster and faster as indicated by P'". While this statement does not accurately capture cases where P' is negative, it does represent the idea that P' can be used to compare how P is changing as x continually increases, namely that P increases "faster and faster." While completing the same task, Student 3 used rate as an indicator for both speed and direction in which P would change over time: "if this number [P'] is really big then in short amounts of time the population goes through a lot of growth. but ... if it was negative then the population would be decaying as time goes on." When completing I4T2, Student 3 noted "*dy/dt* is the derivative of y, its how y is changing at t." In addition he said soon after, "the thing is that since *dy/dt* is the derivative of $y(t)$ it's the slope of, it's basically y over t, it's the slope of y as a function of t." In both cases he unpacked *dy/dt* as an indication of how y was changing at a particular t value. However, the second statement is slightly more complex as he notes that this change depends on t ("the slope of y as a function of t").

While these two ways of interpreting rate are important independently, the students in our study often used them in combination while reasoning about the behavior of solutions. In fact, we feel that being able to *simultaneously* attend to a rate as single quantity *and* unpack it was a critical part of thinking about DEs for these students. Aside from instances when students engaged in the utilization of analytical solution methods, our students often made single statements that indicated both ways of interpreting rate. For instance Student 2 when working with task I1T3 said, "there is a positive correlation between the population of fish and the rate of change. So the more fish there are the faster the fish population will grow." Here, Student 2 reasoned about how P' was changing as P changed (both increase; "a positive correlation"), and simultaneously unpacked rate to determine how P was going to change ("the more fish there are the faster the fish population will grow"). It is important to note that this last determination was more detailed than simply direction or an amount of change in a single instance; the phrase "the faster the fish population will grow" indicates a comparison over various instances. Namely, Student 2 interpreted rate as a single quantity that changed in relation to P, and simultaneously unpacked rate to not only determine the fish population would increase (presumably with respect to changes in time), but that it would increase faster as time changed.

Students' Usage of Covariational and Multivariational Reasoning

General usage of covariational reasoning. Essentially every level of covariational reasoning was used by these three students, suggesting the need for fluency with covariation when reasoning with DEs. The only level not directly observed was *chunky* continuous reasoning, likely because the interview questions were not set up to prompt its usage. Importantly, despite three quantities (t, P, and P') being part of the DEs, the students often focused only on two of these quantities at a time. For example, while discussing $P' = 3P$, Student 2 stated, "As P increases, the slope of P increases." This coordination involved the two quantities directly present in the equation, without explicit attention to t, even though Student 2 had previously recognized its implicit presence. Even with equations with all quantities present, as in $P' = 2P + 2t$, the students often still focused only on two at a time. For example, Student 1 stated, "As t goes up, just, this number [i.e. $2t$] is going to increase, and since it is being added, P' is going to be greater." This coordination involved only t and P' in this statement. It was in a separate instance of reasoning that Student 1 coordinated P and P'.

The importance of recognizing dependence (and non-dependence). In our usage, the first mental action in covariational reasoning is to *recognize dependence* (see Carlson et al., 2002). This may seem quite trivial, as evidenced by the fact that *recognition* is not even explicitly a part of the new Thompson & Carlson (2017) framework. However, a significant part of our students' cognitive efforts in reasoning about DEs involved *recognizing* quantities that may or may not be dependent on each other. For example, when discussing $P' = 3P$, Student 1 stated, "Since P is a function of time, P' is also a function of time." Note that task I1T2 contains no mention of time, nor a variable t. Student 1 *recognized* that such a variable should be *implicitly* present. All three students made such recognitions, where the third variable was often envisioned as time, though Student 2 did acknowledge that it could be "x or t or whatever this P' is taken with respect to."

In the case of DEs, there also appears to be an important parallel mental action to recognizing dependence, wherein students recognize when quantities are *not dependent* on each other. For example, while discussing the equation $P' = 3P$, Student 2 stated that the variable t "is completely irrelevant in terms of the behavior" of the derivative P'. He clarified that a specific solution function $P(t)$ *is* dependent on the variable t, but that the rate of change, P', is not impacted by t. As another example, while Student 3 was working on task I2T4, matching equations to graphs, he explained, "The way this one [the first slope field] didn't change with t, this one [the third slope field] isn't really changing with y." This reasoning action greatly facilitated Students 2's and 3's identification the corresponding DEs on task I2T4.

Multivariational reasoning. While students often discussed only two quantities at a time, they at times engaged in *multivariational* reasoning. All three students invoked *dependent multivariation* by recognizing that t, P, and P' were interdependent quantities. For instance, Student 1 stated, "Since P is a function of time, P' is also a function of time." Sometimes students noted *non-dependence*, as described in the previous subsection, where they recognized that a change in one quantity might not correspond to changes in another quantity, suggesting *independent multivariation*. There was also an occasional use of *nested multivariation*, as seen in the excerpt from Student 2, given earlier, while discussing the equation $P' = 2P + 2t$. He explained that an increase in t first led to an increase in $2t$, which in turn led to an increase in P'. Thus, we can see the nested structure of $t \rightarrow 2t \rightarrow P'$. It appears, then, that multivariational reasoning may be an important aspect of interpreting DEs, in addition to two-quantity covariation.

A New Type of Variational Reasoning: Feedback Variation

In our study, we identified a type of variational reasoning not previously described in the literature. To exemplify, consider Student 3 discussing the equation $P'=3P$: "So, say as P increases, like if P is positive, the rate is positive, so then P would be increasing, and that would in turn increase the rate, then in turn increase P." Later, while discussing the equation $P' = 2P + 2t$, Student 3 also said, "As P changes, it's also affecting its own rate because of this equation." In typical covariation, it is imagined that changes in one variable (x) are related to changes in a *separate* variable (y). However, in this case, Student 3 was explaining how P is related to changes in *itself*. Student 2 made similar statements, by couching P in the real-world context of fish population: "As there is more fish, it supports more growth… If you have more fish, more fish make more fish." Like Student 3, we see Student 2 explaining how a quantity's value dictates how that same quantity will change. It is true that covariation between population and time is implicit, because population cannot change without elapsed time, but the student's focus is on the single quantity P, and how it influences changes in itself. In another task involving y and dy/dt, Student 2 stated explicitly that a DE "is representing what is the effect of [y] with respect to itself." We call this type of variation *feedback variation*, because of how it reminds us of a feedback loop in a microphone/speaker system. In the analogy, the output from the speaker continuously feeds back into the microphone and back out through the speaker, increasing the feedback volume. For P'=3P, one might imagine the speaker to be analogous to P and the microphone to be analogous to P'.

Further, we see a slight nuance to some of the articulations of feedback variation. Notice that in the first excerpt from Student 3, the flow of reasoning is that the quantity P has a value, *then* the rate is positive, *then* the quantity increases, *then* the rate increases, and so on. The language suggests imagining a sequence of discrete steps, similar to what Thompson and Carlson (2017) call *precoordination*. Thus we call this type of reasoning *precoordination of feedback variation*. Of course, the way Student 3 articulated his reasoning may simply be an artefact of attempting to communicate his thoughts to the interviewer. For example, in the second excerpt from Student 3 given above, he explained "As P changes, it's also affecting its own rate." This statement could indicate thinking not of discrete steps, but of a continuously evolving system in which P is always impacting its own rate of change. If a person envisions such a continuously evolving system, we call it *continuous feedback variation*.

New Types of Objects Used in Covariational Reasoning

At one point while discussing a DE involving y and dy/dt, Student 1 drew a graph of a solution function. When the interviewer asked if it was the solution function, Student 1 clarified that it was "one of the potential y of t functions, because there is an infinite [number of them], based on your initial condition." Student 1 recognized that different initial conditions would be associated with different specific solution functions, $y(t)$. Student 2 expressed a similar idea when he stated, "The *y-naught* allows you to put it to a specific situation… Then just literally sliding it [i.e. the graph] over to the point that you need." Here it appeared that Student 2 imagined a *continuously changing* solution graph that ranged over many possible initial conditions until it reached the desired initial condition. In other words, as the initial condition changed, the solution graph changed. Further, Student 3 talked more explicitly about how initial conditions might pair with different solutions. When discussing the task shown in Figure 2, he stated, "So, based on what initial conditions you have, wherever you start on the curve, you are gonna, like, if you start between -2 and 2, the curve will plateau off at 2. If you start below -2 it will plateau off at -2, and above, the curve will plateau off at 2." Despite the incorrect assertion for initial conditions below -2, the point is that he imagined changes in initial conditions leading to changes in the solution function.

These three students appeared to be *covarying* initial conditions and solution functions. Typical covariation usually deals with two numeric quantities, such as x and y (e.g., Carlson et al, 2002; Confrey & Smith, 1995; Johnson, 2015; Moore et al, 2013; Saldanha & Thompson, 2002). However, our students imagined covariation as happening between initial conditions and solution functions, which are different types of objects than discussed in the literature. Of course, depending on the definition of covariation, this may or may not even be considered "covariation," if covariation is only between *numeric-value*-type objects (see Thompson and Carlson 2017, p. 423). However, we suggest it may be appropriate to consider other objects to be *covarying* as we move into more abstract forms of mathematics. Our analysis suggests students have images of initial conditions and solution functions as changing (varying) together (co). Because these objects do not have "values" in the same way as numeric quantities do, some mental actions like coordination of values might not have matches for this context. However, Student 2 may even have employed *continuous covariational* reasoning by imagining a graph sweeping through initial conditions until the desired initial condition was reached.

Discussion

Our results show that variational reasoning is important for unpacking and understanding DEs. Our results further indicate that there may be unique aspects to variational reasoning for DEs. A key part of our students' mental work was recognizing what quantities are implicitly contained in a DE, what quantities are dependent on each other, and what quantities are *not* dependent on each other. This greatly expands the *recognize dependence* mental action in Carlson et al.'s (2002) original framework, and underscores its importance. Thus, we suggest that it should not be dropped from the new covariational framework (Thompson & Carlson, 2017), but be incorporated as an important skill students may need as they advance to more complicated mathematics. Additionally, we have identified a new type of variational reasoning outside of current covariation and multivariation frameworks (Carlson et al., 2002; Jones, 2018; Thompson & Carlson, 2017). DEs have a unique structure wherein a quantity is explicitly related to changes *in itself*. In other words, its current value indicates how it will change. This does not occur in covariation between, say, x and y, where x is free to vary as a, literally, *independent* variable. Finally, we saw that students appeared to employ covariational reasoning with new types of objects beyond what it typically described in the literature. In addition to covarying numeric quantities, the students covaried points (initial conditions) and functions (particular solutions). There are even likely different levels to covarying these types of objects. One could imagine a change in initial condition *then* a change in particular solution (*precoordination*), a generic imagine of the initial condition moving to the right as the solution function changes in some way (*gross coordination*), or one could imagine a "sweeping" initial condition with specific values that continuously passes through infinitely many specific particular solutions (*continuous covariation*).

Our work also illuminates the importance of reasoning with rate as both a single quantity and as a relationship between two varying quantities when making sense of DEs and their solutions. While the covariation literature discusses constructing rate by composing amounts of change in two related quantities, much of our students' mental work consisted of decomposing rate. That is, they reasoned with the DE as if it were a function to understand how the values of the various quantities changed, but then also unpacked the rate to make sense of how the intrinsic quantities behaved. They used both notions of rate to conceptualize the solution functions. They took a rate as a single, changing quantity, decomposed it into two quantities and used it to perceive the relationship between the two quantities so that they could construct a solution function. Further, they often simultaneously coordinated changes in the rate (as indicated by the DE itself) with changes in the two quantities from which it was composed. Our findings regarding students' frequent utilization of variational reasoning and the various ways of working with rate align with and add to the thematic nature of function and rate of change in the research literature on DEs. For instance, our findings bring together and strengthen Donovan's (2007) assertion regarding the importance of conceptualizing a DE as a function, Keene's (2008) work regarding student reasoning with rate of change, and Whitehead and Rasmussen's (2003) discussion of rate use. In this case, examining how the students reasoned about relationships between varying quantities allowed us to understand some of the ways these two concepts come together when reasoning about DEs.

This work suggests it is important for instructors to provide students with opportunities in which they engage in reasoning with DE's in two ways: as a relationship between bare variables, and as a relation between the value of a function and its corresponding rate of change at a particular instance. The latter may require focused and meaningful attention on the often implicit inclusion of the functions independent variable. Importantly, instructors must get students to consider both of these relationships simultaneously.

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