Characterizing Conceptual and Procedural Knowledge of the Characteristic Equation

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Research on student understanding of eigentheory in linear algebra has expanded recently, yet few studies address student understanding of the Characteristic Equation (CE). In this study, we explore students' conceptual and procedural knowledge of deriving and using the CE. Consulting Star's (2005) characterization of deep and superficial conceptual and procedural knowledge, we developed the Conceptual and Procedural Knowledge framework for classifying the quality of students' conceptual and procedural knowledge of both deriving and using the CE along a continuum. Most of our students exhibited deeper conceptual and procedural knowledge of using the CE than of deriving the CE. Furthermore, most students demonstrated deeper procedural knowledge than conceptual knowledge of deriving the CE. Examples of student work are provided, and implications for instruction and future research are discussed.

Keywords: procedural knowledge, conceptual knowledge, linear algebra, eigentheory

Considering recent demands for enhanced student understanding of concepts in science, technology, engineering, and mathematics fields, education researchers are tasked with exploring how students make sense of mathematical concepts in interdisciplinary settings. Our study focuses on quantum physics students' understanding of eigentheory in linear algebra. Eigentheory encompasses topics related to eigenvalues, eigenvectors, and eigenspaces. Students encounter eigentheory in a variety of contexts and courses, such as linear algebra, differential equations, numerical analysis, and quantum physics. Thus, it is essential for researchers to examine student understanding of eigentheory due to its interdisciplinary nature. This situates our work that investigates how students reason about and symbolize eigentheory in linear algebra and in quantum physics (*Project LinAl-P*, NSF-DUE 1452889).

A central tool often used to calculate the eigenvalues of an $n \times n$ matrix A is the *characteristic equation* (CE) of A, defined as $det(A - \lambda I) = 0$, for an $n \times n$ identity matrix I and scalar λ. The determinant of $A - \lambda I$ gives the characteristic polynomial of A, and the roots of this polynomial are the eigenvalues of A. In addition to symbolically representing a procedure, the CE is conceptually related to the Invertible Matrix Theorem (IMT, Lay, 2003). The CE can be derived from the eigenequation $Ax = \lambda x$, by subtracting λx from both sides ($Ax - \lambda x = 0$), introducing the identity matrix to get the homogeneous equation $(A - \lambda I)x = 0$, and making connections to the IMT. For example, one can recognize that for the equation $(A - \lambda I)x = 0$ to yield more than just the trivial solution for x (as eigenvectors cannot be the zero vector), the matrix $A - \lambda I$ must not be invertible, which implies that the determinant of $A - \lambda I$ must be zero.

Instructors want their students to conceptually understand these connections between the CE, the IMT, and related eigentheory concepts, yet some researchers posit that students struggle to do so (e.g., Bouhjar, Andrews-Larson, Haider, & Zandieh, 2018). Bouhjar et al. claimed:

> There is a disconnect between students' understanding of standard procedures for finding eigenvalues and the formal definition of an eigenvector and eigenvalue, and… students

are more able to execute the standard procedure than draw on conceptual understandings aligned with the formal definition. (p. 213)

This disconnect seemed apparent in our own interview data with quantum physics students regarding their understanding of eigentheory. Although most of the students participating in our study successfully determined the eigenvalues of a given 2×2 matrix during an interview task, several students volunteered, sometimes unprompted, that they did not know why the CE is used or is true. When discussing why the determinant of $A - \lambda I$ must be zero to solve for the eigenvalues λ of A, some of the students explained, "That's just what I was taught," and the CE is true "because of something in linear algebra that says it needs to be this way." Another student explicitly expressed this focus on the procedure: "I remember learning why [using the CE] is the thing that I do. But… if I ever encounter a problem where I need eigenvalues, like, this is the first thing that comes to mind and not like where that comes from." This emphasis on the procedural use of the CE in our interview data led us to explore students' conceptual and procedural knowledge of the CE. We address the following research question: How do quantum physics students reason with and about the CE?

Literature Review

Various research studies (e.g., Boujar et al., 2018; Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Plaxco, Zandieh, & Wawro, 2018; Salgado & Trigueros, 2015; Thomas & Stewart, 2011) have explored student understanding of eigenvalues, eigenvectors, and related concepts, yet we have not found any that specifically focus on characterizing students' understanding and use of the CE. Thomas and Stewart (2011) focused on how students interpreted $Ax = \lambda x$, finding many students were comfortable with the procedural algebraic manipulations of matrices and vectors, as in Tall's (2004) symbolic world, but the students did not hold embodied ideas regarding eigenvalues and eigenvectors. They asserted that students' fluency in symbolic manipulations should be paired with understanding what the symbols represent. In particular, they pointed out the importance of understanding the resulting product on both sides of the equation $Ax = \lambda x$ is the same vector and understanding why the identity matrix is used in transitioning from $Ax = \lambda x$ to $(A - \lambda I)x = 0$, which many students in their study struggled to articulate.

Other studies demonstrated students' rich understanding of connections between concepts related to eigenvalues and eigenvectors (e.g., Larson, Rasmussen, & Zandieh, 2008; Salgado & Trigueros, 2015; Wawro, 2015). Wawro (2015) exemplified a student who made connections between statements in the IMT by reasoning about solutions to matrix equations, span, linear independence, null space, and the eigenvalue zero. Larson, Rasmussen, and Zandieh (2008) highlighted one student's ability to make connections between linearly dependent column vectors and the zero determinant of a matrix by reasoning about determinant graphically as the area of a parallelogram formed by two column vectors. More directly related to student understanding of the CE, Salgado and Trigueros (2015) described the reasoning of a group of three students who derived the CE without prior instruction by making connections to statements in the IMT, demonstrating conceptual understanding needed to reinvent the CE on their own.

Most relevant to our current study, Bouhjar et al. (2018) characterized students' responses to an open-ended written question that asked if 2 was an eigenvalue of a given 2x2 matrix. The authors claimed students who reasoned about the determinant used a more procedural approach, and students who reasoned about the matrix $A - \lambda I$ without computing the determinant used a more conceptual approach, as characterized by Hiebert and Lefevre's (1986) definitions of conceptual and procedural knowledge. However, the authors described their difficulty in classifying written work as demonstrating conceptual or procedural knowledge of the CE:

> It was often unclear from the responses of students who used the standard procedure [seeing if $\det(A - 2I) = 0$] whether they understood links among the equation used in defining eigenvectors, the solution set of $(A - \lambda I)x = 0$, and the equivalencies in the invertible matrix theorem that lead to use of the determinant as a tool for determining when the solution is non-trivial. (p. 212)

Furthermore, since many students simply used the CE to find the eigenvalues of A directly with no other explanation, the authors were unable to explore those students' conceptual understanding of derivation of the CE. Bouhjar et al. claimed more work is needed to distinguish whether a student using the CE to find eigenvalues just uses the procedure by rote or actually has deep conceptual understanding of why the CE works. Our analysis of students' interview responses about the derivation and use of the CE contributes toward this need by characterizing, along a continuum, students' conceptual and procedural knowledge in this context.

Theoretical Background

Conceptual Knowledge (CK) and Procedural Knowledge (PK) are qualitative constructs commonly used by mathematics education researchers to classify students' mathematical knowledge. Hiebert and Lefevre (1986) defined CK as "knowledge that is rich in relationships… a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3-4). They defined PK as "familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols" (p. 7), which consist of "rules or procedures for solving mathematical problems" (p. 7). Star (2005) argued that these definitions conflate students' *type* of knowledge with *quality* of knowledge, as if PK could never be as rich in connections as CK. Star further argued that holding CK is not necessarily better than PK; rather, both types of knowledge are essential for consummate understanding of mathematics. Thus, Star (2005) proposed classifying knowledge according to both quality (either deep or superficial) and type (either procedural or conceptual). He defined *deep PK* as "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment" (p. 408). A student demonstrates deep PK when (s)he can provide a "cogent explanation of how the steps are interrelated to achieve a goal" (Baroody, Feil, & Johnson, 2007, p. 119). *Superficial PK* is knowledge of procedures that is not richly connected (Star, 2005). Star characterized *deep CK* as richly connected knowledge of concepts, and *superficial CK* as knowledge of concepts that is not richly connected.

Classifying students' knowledge quality as deep or superficial can seem quite extreme, given that not all students exhibit strictly deep or superficial CK and PK. Therefore, we propose including a moderate knowledge quality as a classification for students who demonstrate deeper knowledge than students exhibiting superficial knowledge, yet less deep knowledge than those exhibiting deep CK or PK. We offer a framework for characterizing aspects of students' Superficial, Moderate, and Deep CK and PK of the CE, as described in the Methods section.

Methods

The data for this study consist of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 17 students enrolled in a quantum mechanics course. The interviews occurred during the first week of the course. Nine of the students were from a junior-level course at a large public research university in the northwest United States (school A), and the other eight were in a senior-level course at a medium public research university in the northeast United States (school C). All students are pseudonymed with "A#" or "C#." Interview questions aimed to elicit student understanding of several linear algebra concepts which they would use in the quantum mechanics course.

For this paper, we focus on students' attempts to recall, derive, and/or use the CE within their response to one particular interview question. Students were first asked, "Consider a 2×2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. How do you think about $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?" After follow-ups inquiring if they had a geometric or graphical way to think about the equation and how they thought about the equal sign in this context, students were asked how they thought about the equation if $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and to determine values of x and y that would make the equation true. Finally, students were asked to find the eigenvalues and eigenvectors of A, if they had not already done so. Note the interview question was designed so the terms "eigenvector" and "eigenvalue" were not used until the end; however, many students immediately recognized the first matrix equation as an eigenequation and often brought up eigentheory ideas on their own.

To begin our analysis, we wrote detailed descriptions for each student of their work on the aforementioned interview task, focusing on their reasoning about the CE. These descriptions contained evidence from the transcripts and images of students' written work. Using these descriptions, and consulting Star's (2005) definitions of deep and superficial CK and PK, we began to develop the Conceptual and Procedural Knowledge (CPK) framework for the CE (see Figure 1). The CPK framework describes characteristics of student work demonstrating both CK and PK across two dimensions: deriving the CE and using the CE. Through discussing the knowledge qualities demonstrated by the students in the context of the eigenvalue task, we developed lists of characteristics of student work demonstrating Superficial, Moderate, and Deep PK and CK for both deriving and using the CE. These lists were revised and organized into the CPK framework, which was used to code each student's response to the eigenvalue task.

	N/A	*Superficial*	*Moderate*	*Deep*
Procedural Knowledge of Deriving the CE	Does not attempt to write the CE	Incorrectly writes the CE (e.g., $A - \lambda I = 0$) and does not attempt to explain the symbolic derivation of the CE	Attempts to write the CE and make connections between $Ax = \lambda x$, $(A - \lambda I)x = 0$, and $\lvert A - \lambda I \rvert = 0$, but uses symbols incorrectly	Accurately manipulates symbols among $Ax = \lambda x$, $(A - \lambda I)x = 0$, and $\lvert A - \lambda I \rvert = 0$ to derive the CE, and writes the CE correctly
Conceptual Knowledge of Deriving the CE	Does not attempt to explain how the CE is derived	States they do not know where the CE comes from or gives irrelevant explanation of how the CE is derived	Gives explanation of how the CE is derived from $(A - \lambda I)x = 0$ that is relevant to the IMT, yet incorrect	Accurately explains how the CE is derived from $(A - \lambda I)x = 0$, while referencing connections to the IMT
Procedural Knowledge of Using the CE	Does not use the CE procedure to find eigenvalues	Correctly uses the CE procedure to find eigenvalues, without exhibiting fluency in algebraic manipulations or rigor in the calculations	Correctly uses the CE procedure to find eigenvalues, while exhibiting either fluency in algebraic manipulations or rigor in the calculations	Correctly uses the CE procedure to find eigenvalues, while exhibiting both fluency in algebraic manipulations and rigor in the calculations
Conceptual Knowledge of Using the CE	Does not recognize eigenvalues are the results of using the CE and does not use or discuss the resulting eigenvalues in the context of other eigentheory concepts	Recognizes eigenvalues are the results of using the CE but does not use or discuss the resulting eigenvalues in the context of other eigentheory concepts	Recognizes eigenvalues are the results of using the CE and makes only one connection between the eigenvalues resulting from the CE and other eigentheory concepts; OR makes two or more connections with at least one being incorrect	Recognizes eigenvalues are the results of using the CE and correctly makes two or more connections between the eigenvalues resulting from the CE and other eigentheory concepts.

Figure 1. Conceptual and Procedural Knowledge (CPK) framework for the CE

We now briefly explain each of the four rows of the framework. PK of deriving the CE entails symbolically moving from the eigenequation $Ax = \lambda x$ to the homogeneous equation $(A - \lambda I)x = 0$, and introducing the determinant to get $|A - \lambda I| = 0$. CK of deriving the CE involves making connections to the IMT to explain why the determinant of $A - \lambda I$ must be zero. PK of using the CE involves knowing the CE is an appropriate procedure to use to find eigenvalues and demonstrating fluency (i.e., ease of carrying out calculations) and rigor (i.e., making sure to write "= 0" at each step) in employing the CE. CK of using the CE entails understanding that the solutions of the CE are eigenvalues and making connections to other aspects of eigentheory (e.g., recognizing that the found eigenvalue 2 is the same 2 as in the original equation, plugging the found eigenvalues into $Ax = \lambda x$ or $(A - \lambda I)x = 0$ to attempt to find the eigenvectors, explaining what the found eigenvalues mean geometrically). For this last row, it is important to note that only the correctness of the connection was judged (e.g., plugging the eigenvalue into a correct equation like $Ax = \lambda x$ or $(A - \lambda I)x = 0$), not their knowledge of finding eigenvectors, or even what the equations in eigentheory mean. In the Results section, we explain how this framework helped us gain further insight into students' CK and PK of the CE.

Results

Responses of 3 of the 17 participating students were coded as "N/A" in all four categories, and one was coded as "N/A" in all but one category; thus, we focus our Results section on analyzing the remaining 13 students. Our four-part theoretical framework allowed us to unpack different aspects of students' understanding of the CE. The number of students exhibiting N/A, Superficial, Moderate, and Deep knowledge in each of the four categories is provided in Table 1. We share three prominent results from our analyses in the remainder of this section.

Table 1. Number of students exhibiting N/A, Superficial, Moderate, and Deep PK and CK of the CE

	N/A	Superficial	Moderate	Deep
PK of Deriving the CE	0	3	8	2
CK of Deriving the CE	0	10	2	1
PK of Using the CE	1	2	5	5
CK of Using the CE	1	1	3	8

First, our CPK Framework for the CE illuminated that three students (A8, A11, and C5) showed relatively high sophistication in using and deriving the CE. In particular, two students demonstrated Deep knowledge in three of the four areas and moderate knowledge in a fourth, and another student showed deep knowledge in two areas and moderate knowledge in the other two categories. Taking A8 as a particular example, he first manipulated $Ax = \lambda x$ cleanly into $(A - \lambda I)x = 0$ (see Figure 2a), demonstrating Deep PK of deriving the CE. He then correctly stated there is a nonzero solution for x when $A - \lambda I$ is singular, connecting the CE to the IMT and exhibiting Deep CK of deriving the CE. When using the CE, A8 correctly calculated the eigenvalues with no apparent difficulty. However, his notation was somewhat improper, manipulating the polynomial in the CE by itself rather than as an equation (see Figure 2b). For this reason, we coded this as showing Moderate PK (partially due to this omission being associated with other common student errors in factoring). After finding the eigenvalues, A8 showed Deep CK by making connections both to finding eigenvectors using $(A - \lambda I)x = 0$ and to the previous part of the problem, where the eigenvalue 2 was given in an eigenequation.

Second, our data showed that the students in our study were better at using the CE than deriving the CE, both procedurally and conceptually. Students' CK of using the CE seemed

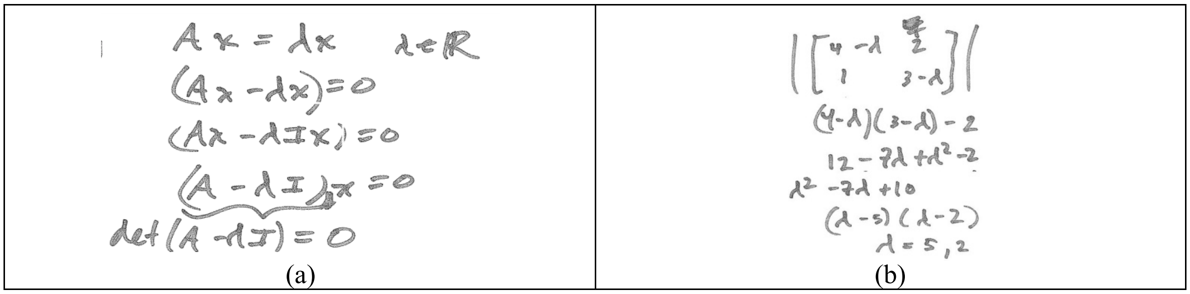

Figure 2. (a) A8's symbolic derivation of the CE; (b) A8's use of the CE

stronger than their CK of deriving the CE, seen as 10 out of 13 students exhibited Superficial CK of deriving the CE, but only 1 out of 13 students exhibited Superficial CK of using the CE. Also, only 1 of 13 students exhibited Deep CK of deriving the CE, but 8 out of 13 exhibited Deep CK of using the CE. C3 exemplified this trend of exhibiting deeper CK of using the CE than of deriving the CE, as he demonstrated Superficial CK of deriving the CE and Deep CK of using the CE. In particular, when asked to find the eigenvalues and eigenvectors of A, C3 first wrote an appropriate homogeneous equation (Figure 3a), crossed out the "equals zero," and said it was the determinant of that which equaled zero (Figure 3b). He then explained he could cross out the $\begin{bmatrix} x \\ y \end{bmatrix}$ "because you're dividing it out," claiming the vectors in the eigenequation $Av = \lambda v$ cancel (see Figure 3c). Once C3 found 2 and 5 as the eigenvalues of A, he mentioned "you could have given me 5," in reference to the original $Ax = 2x$ equation, and he used $Ax = 5x$ to find other eigenvectors. Even though C3 did not seem to figure out a conceptual derivation of the CE, he recognized the CE solutions as eigenvalues and made connections between those and the eigenequation to find eigenvectors. This exemplar illustrates our result that our students connected the CE with eigentheory concepts, but they did not seem to know why the CE is true.

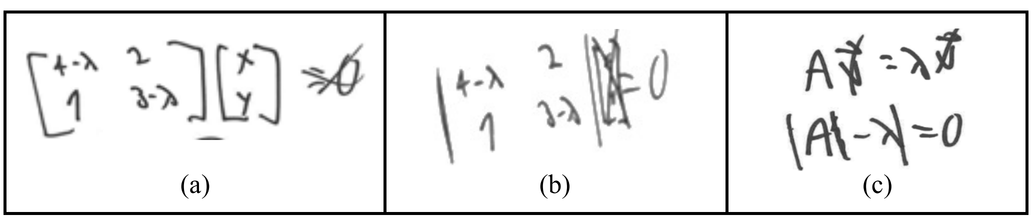

Figure 3. C3's written work for his derivation of the CE

Furthermore, students in our study seemed to have stronger PK of using than of deriving the CE (see Table 1). Most students wrote the CE correctly or made small mistakes writing it, but did not accurately make connections between equations like $Ax = \lambda x$, $(A - \lambda I)x = 0$ and $|A - \lambda I| = 0$. However, most students had little difficulty in using the CE to find eigenvalues. C3 exemplified this trend because he demonstrated Deep PK of using the CE and Moderate PK of deriving the CE. C3's symbolic manipulations (see Figure 3) revealed he could not accurately derive the CE from the eigenequation. Nevertheless, he fluently and rigorously used the CE.

Lastly, in comparing students' PK to their CK within both dimensions, contrasting trends emerged. In deriving the CE, all students demonstrated PK that was as deep or deeper than their CK. In some ways, this is not surprising as many students (10 of the 13) did not make any connection to the IMT in their explanation of deriving the CE, but most (10 of the 13) wrote the CE correctly and/or made connections to $Ax = \lambda x$ or $(A - \lambda I)x = 0$. By contrast, looking at using the CE, a majority of students (11 of the 13) demonstrated CK that was as deep or deeper than their PK. Looking back at A8 as a particular example, recall that he fluently found eigenvalues and connected them back to both the homogeneous equation and the equation given in the initial problem statement (demonstrating Deep CK for using the CE), but did not rigorously write "$= 0$" after each step in the calculations (demonstrating moderate PK for using the CE). We recognize this trend between PK and CK in using the CE is largely a result of the choices we made on characterizing "deep knowledge" within each dimension. In particular, we note that categorizing students who do not rigorously write "$= 0$" as having Moderate PK in using the CE (such as A8), and categorizing students who correctly connected the found eigenvalues to other eigentheory elements as having Deep CK in using the CE, regardless of their abilities to find eigenvectors or explain what eigenvalues mean, are subjective decisions. However, we feel our analysis highlights that many students *do* know how to find the values of λ that make $|A - \lambda I| = 0$ true, despite work that appears non-rigorous, and understand how this process produces the eigenvalues, which are essential to all other aspects of eigentheory.

Conclusion

Using the CPK framework to code students' interview responses allowed us to distinguish students' type and quality of knowledge of both using and deriving the CE. We captured student understanding of deriving the CE, which was not accessible in the written data in Bouhjar et al.'s (2018) study; hence, our work addresses their call to determine whether students employing the CE to find eigenvalues only know how to use the procedure or also understand how it works. We found the students in our study were better at using the CE than deriving it. Most students experienced little to no difficulty in using the CE to find eigenvalues and making connections to other eigentheory concepts, but they seemed to struggle with knowing how it is derived conceptually. To address this lack of Deep CK, instructors could intentionally enhance students' understanding of the IMT and help them make connections to it while deriving the CE. Instructors could also emphasize precision in symbolically deriving the CE to help students learn how to accurately manipulate symbols associated with eigentheory concepts.

This study offers a theoretical contribution regarding the addition of the classifications N/A and Moderate to delineating the quality of PK and CK; these allow for finer nuance in classifying the quality of students' CK and PK. In the CPK framework, we also offer the distinction of student understanding of deriving and using the CE to provide more insight into how students think about these different aspects. The CPK framework can be generalized for investigating student understanding of topics in linear algebra and other areas of mathematics, but the characteristics of student work listed in each cell of the framework may change, depending on the mathematical content and the nature of the tasks students perform. This framework seems most useful for analyzing student interview data, since interviewers can prompt students to both perform procedures and explain their thinking about concepts. To use the CPK framework with written data, the written tasks should elicit evidence of students' reasoning about both the derivation and use of the mathematical topic. Further research can explore how bidirectional relations form between CK and PK, as proposed by Rittle-Johnson and Schneider (2015), focusing on the how students' CK of the CE supports their PK of the CE, and vice versa.

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