Analyzing Students' Understanding of Isomorphism

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The purpose of this study is to analyze student understanding of isomorphism as it is taught in a university level mathematics course. We collected and studied student responses to course assignments covering the concept of isomorphism. The findings of this study support previous research that suggests student understanding of isomorphism is largely reliant on an imaged-based concept of symmetry. We found that student understanding is supported by an image-based radical constructivist approach and detail the techniques students use when first working with isomorphic mappings.

Keywords: Isomorphism, Abstract Algebra, Qualitative Methods

The concept of isomorphism is a core component of all Abstract Algebra courses. It holds the power to traverse mathematical operations and meaning between different groups and between different realms of mathematics. Mena-Lorca and Parraguez (2016) describe it as a "difficult" concept for undergraduate students (p. 378). This is partly because the concept of isomorphism builds from multiple other concepts in mathematics. To understand isomorphism, one must first have thorough knowledge of functions, one-to-one correspondence, groups, and homomorphism (Pinter, 1990). Furthermore, understanding isomorphism is significant because it requires a high level of abstract thinking that is not often reached in lower level mathematics courses (Larsen, 2013). In this way, understanding isomorphism transitions students from lower level mathematics concepts to more advanced concepts. Altogether, a strong comprehension of isomorphism can equip students to successfully study group theory and other theoretical mathematics topics. Hence, there is a substantial need for analysis of how students understand the concept of isomorphism. This research seeks to gain knowledge of student understanding of isomorphism as it is introduced in an upper-level university mathematics course that practices radical constructivism. The purpose of this study is to describe how students develop an understanding of isomorphism to improve the quality and effectiveness of undergraduate mathematics education.

Background Literature

Relatively little attention has been given to teaching methodologies that aim to minimize the void between confusion and understanding for undergraduates studying upper-level mathematics. Moreover, almost no research is dedicated to the study of how students understand elementary Group Theory topics such as isomorphism. It has been the opinion of current researchers that "the teaching of abstract algebra cannot be considered a successful endeavor" because students must work with unfamiliar, abstract concepts when they have previously relied on strict, procedural proof techniques (Mena-Lorca & Parraguez, 2016, p. 378). The most recent studies of this topic (Mena-Lorca & Parraguez, 2016; Larsen, 2009, 2013) seek to address how students' understanding of isomorphism stems from their pre-existing informal knowledge.

A case study teaching experiment of two students investigated how students could reinvent the ideas of groups and isomorphism using pre-existing knowledge (Larsen, 2009).

Larsen’s guided reinvention approach used basic concepts such as the symmetries of an equilateral triangle to support student discovery. This study identified informal student strategies used to grasp the concepts at hand and suggested how these strategies could be evoked to support the reinvention process and learning of formal concepts (Larsen, 2009). In a similar study, Larsen (2013) formed a series of design experiments to support the reinvention approach to teaching group and isomorphism concepts. Most recently, a large-scale study published in 2018 captured a representative, nation-wide sample of student responses while working with the concepts of subgroups, cyclic groups, and isomorphism. This study expanded previously conducted, non-representative studies, establishing the expanse of different student conceptions and re-analyzing current theories (Weber, 2001; Weber & Alcock, 2004) on student understanding of isomorphism, suggesting that students take a slightly more semantic approach when working with isomorphism than once perceived. That is, in the study, students tended to explore groups structurally rather than within the formal definition when determining isomorphism (Melhuish, 2018).

Methods

Participants

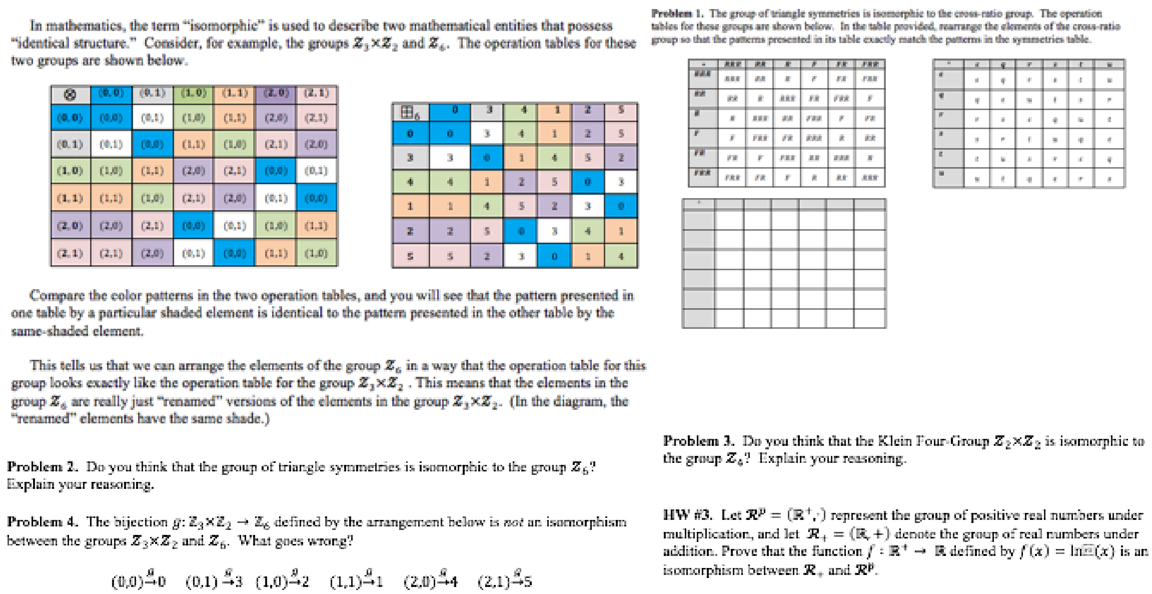
The participants of the study consisted of students majoring in mathematics or mathematics education enrolled in an Abstract Algebra I course at a southeast university. Data was collected from a total of 19 students in two classes over two semesters. Abstract Algebra I is considered the first upper-level mathematics course for the participants hence, these students had no previous course study in upper-level mathematics topics such as Analysis, Graph Theory, or Number Theory that may also cover types of isomorphic structures and relationships.

Task/Context

The instructor of the course utilizes a radical constructivist approach (Glaserfeld, 1995) as the learning through to develop a set of materials called, *Pathways to Abstract Algebra*. These materials view the classroom as a place for exploration of concepts through creating conjectures and making discoveries. The role of the instructor is to create learning situations in which this exploration can happen. In class, students work on investigations covering basic Group Theory topics in groups of two to five students. The instructor facilitates and monitors small group discussion, periodically leading full class discussion over questions and tasks in the investigation being completed. This study focuses on the isomorphism investigation. This investigation is designed to allow students to develop an intuition that motivates the properties of isomorphism.

This investigation begins with tasks that prompt students to use previously learned concepts and rudimentary skills such as matching to construct their own understanding of isomorphism. Initially, the students are encouraged to reason from the perspective of “labeling” groups as a way of motivating the function-based definition. Students are shown an example of two isomorphic groups, $Z_3 \times Z_2$ and Z_6, along with their corresponding, color-coded operation tables. In problem 1, students are asked to recreate similar corresponding tables for the group of triangle symmetries and the cross-ratio group. Through this exercise, students should form a visual relationship between the given isomorphic groups. After students gain a mental picture of isomorphism through this exercise, they work on questions that help winnow away false strategies that they may be using to determine if two groups are isomorphic to each other (i.e. exhaustively checking arrangements). Problem 2 asks students to determine if the triangle symmetries group and Z_6 are isomorphic and to explain why they come to their conclusion.

Problem 3 similarly asks students to determine if the groups $Z_2 \times Z_2$ and Z_4 are isomorphic and why. These questions give students the opportunity to recognize reoccurring properties of isomorphism that have not yet been revealed in the investigation.



In mathematics, the term "isomorphic" is used to describe two mathematical entities that possess "identical structure." Consider, for example, the groups $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$. The operation tables for these two groups are shown below.

⊗	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)
(1,0)	(1,0)	(1,1)	(2,0)	(2,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(2,1)	(2,0)	(0,1)	(0,0)
(2,0)	(2,0)	(2,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2,1)	(2,1)	(2,0)	(0,1)	(0,0)	(1,1)	(1,0)

$\boxplus_6$	0	3	4	1	2	5
0	0	3	4	1	2	5
3	3	0	1	4	5	2
4	4	1	2	5	0	3
1	1	4	5	2	3	0
2	2	5	0	3	4	1
5	5	2	3	0	1	4

Compare the color patterns in the two operation tables, and you will see that the pattern presented in one table by a particular shaded element is identical to the pattern presented in the other table by the same-shaded element.

This tells us that we can arrange the elements of the group $\mathbb{Z}_6$ in a way that the operation table for this group looks exactly like the operation table for the group $\mathbb{Z}_3 \times \mathbb{Z}_2$. This means that the elements in the group $\mathbb{Z}_6$ are really just "renamed" versions of the elements in the group $\mathbb{Z}_3 \times \mathbb{Z}_2$. (In the diagram, the "renamed" elements have the same shade.)

Problem 1. The group of triangle symmetries is isomorphic to the cross-ratio group. The operation tables for these groups are shown below. In the table provided, rearrange the elements of the cross-ratio group so that the patterns presented in its table exactly match the patterns in the symmetries table.

Problem 2. Do you think that the group of triangle symmetries is isomorphic to the group $\mathbb{Z}_6$? Explain your reasoning.

Problem 3. Do you think that the Klein Four-Group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the group $\mathbb{Z}_4$? Explain your reasoning.

Problem 4. The bijection $g: \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ defined by the arrangement below is *not* an isomorphism between the groups $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$. What goes wrong?

$$(0,0) \xrightarrow{g} 0 \quad (0,1) \xrightarrow{g} 3 \quad (1,0) \xrightarrow{g} 2 \quad (1,1) \xrightarrow{g} 1 \quad (2,0) \xrightarrow{g} 4 \quad (2,1) \xrightarrow{g} 5$$

HW #3. Let $\mathcal{R}^p = (\mathbb{R}^+, \cdot)$ represent the group of positive real numbers under multiplication, and let $\mathcal{R}_+ = (\mathbb{R}, +)$ denote the group of real numbers under addition. Prove that the function $f: \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$ is an isomorphism between $\mathcal{R}_+$ and $\mathcal{R}^p$.

Figure 1. Problems 1, 2, 3, 4 and HW #3 of the Isomorphism investigation

Following this process, students are introduced to the formal definitions of operation preserving functions, homomorphism, and isomorphism. These definitions are presented through a mini-lecture with the goal to show students that the assignments they have been using to determine isomorphism, in fact, correspond to a function. Students then work on questions that serve to clarify their current understanding of the definition. Problem 4 defines an arrangement between $Z_3 \times Z_2$ and Z_6 that is not an isomorphism and asks students to determine "what goes wrong" within the given assignment. Next, the preservation of identity and inverses property of isomorphism is finally presented as a theorem. Homework problem 3 asks students to prove that a given function is an isomorphism between $\mathcal{R}_+$ and $\mathcal{R}^p$. While this problem does not cover new information, it is a significant indicator of what understanding the students gain from class learning and discussion. Given the structure and goals of these materials, we aim to answer the following research question: how do students develop an understanding of isomorphism?

Data Collection and Analysis

We collected all assignments that included the topic of isomorphism. This includes in-class assignments, homework assignments, quizzes, and exams. We also audio recorded class and small group discussions during the isomorphism investigation. However, for the purpose of this paper we will focus on their written and audio responses to the investigation and homework. All data was blinded and given pseudo-names for analysis.

The problems from which we analyzed data mirror the problems described in the methods section. While students' written work is shown, this is only to give readers a visual idea of student responses; the data analyzed is more extensive than the work displayed in this section and includes recorded discussion. We chose not to analyze data from 7 students because they had previously taken Abstract Algebra I, chose to not be recorded, or were absent during the isomorphism investigation, resulting in a total of 11 students' data.

To describe students' understandings of isomorphism, data was analyzed qualitatively, open-coding for the different aspects of isomorphism, to identify understandings and misunderstandings (Creswell, 2007). We double coded all student responses to each question detailed in the methods section and resolved disagreements through discussion. Specifically, we focused on the techniques through which students completed each problem (i.e. creating operation tables, checking for certain properties). We examined the codes and determined themes of student understanding of isomorphism (i.e., table reliant understanding). These themes are detailed in the results section.

Results

The goal for Problem 1 is for students to develop an intuition for the meaning of isomorphism between two groups and informally recognize properties of isomorphism, such as the preservation of identities and inverses. Altogether, students should focus on the structure of the triangle symmetries and cross-ratio groups rather than the label of each individual element in these groups. Upon analysis, data showed that 91% (n=10) of students successfully found an isomorphic mapping between the triangle symmetries and cross-ratios groups. The same 91% of students began problem 1 by identifying the identity elements of each group and mapping them to each other. An example is shown in Figure 2. Here the elements RRR and ε are first circled in each table and then the ε's are positioned in the bottom table to match the placement of the circled RRR's in the triangle symmetries table. Then the assignment $\varepsilon = RRR$ is made (while this notation is incorrect, the students have not yet been introduced to correct notation).

Of this 91% (n=10) of students who began by identifying the identity elements, five students moved on to map self-inverting elements to each other, three students moved on to mapping non-self-inverting elements, and two students were unable to make more progress and began randomly guessing full mappings. The ten students who were able to make progress and complete the isomorphic mapping tended to assign colors or shapes to the elements they mapped to each other, mimicking the example set forth by the instructor at the beginning of the investigation. Suzie's work in Figure 2 demonstrates this by her markings in the given table of triangle symmetries. Finally, about one third of the students recognized that there are multiple isomorphic arrangements between the group of triangle symmetries and the cross-ratio group.

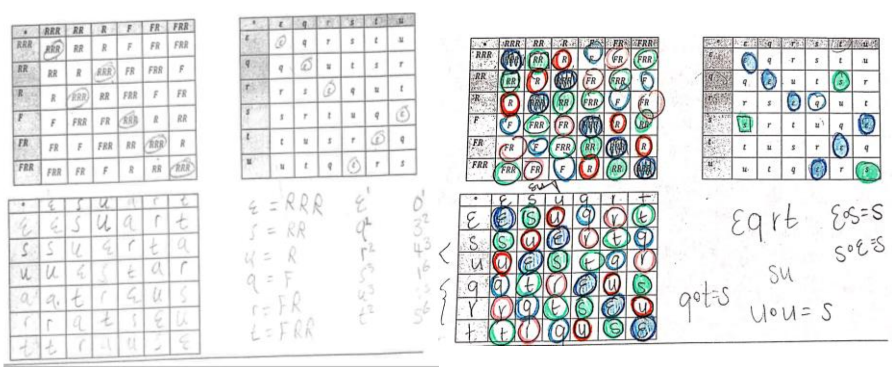


Figure 2. William's (left) identification of the identity elements and Suzie's (right) use of colors

Ultimately, almost all of the students started by mapping identity elements and then self-inverting or non-self-inverting elements, suggesting that students were able to informally recognize the preservation of identities and inverses within an isomorphic mapping. Moreover, two students recognized that there are multiple isomorphic mappings between the group of triangle symmetries and the cross-ratio group, suggesting they started to develop a greater intuition for the meaning of isomorphism.

Problem 2 is designed to help students create a distinction between their mental picture and the isomorphism properties they found in problem 1. This problem aims to refine students' mental image of isomorphism by helping them see what it is not. That is, problem 2 establishes that there is more to the concept of isomorphism than simple matching; isomorphism is, in fact, centered around the structure of the groups. We found that more than half of the students reasoned that the triangle symmetries group and $\mathcal{Z}_6$ group are not isomorphic because they do not have the same number of self-inverting elements. Of these students who recognized the different number of self-inverting elements, all but one did not draw or create their own table to come to this conclusion. For example, Samantha states in her answer, "No, the number of times the identity appears across the diagonal is not the same," meaning she found different numbers of occurrences of the identity in the diagonals of the operations tables for each group.

Contrastingly, the 45% (n=5) of the students did not recognize the different number of self-inverting elements and drew tables for each group, reasoning that they were not isomorphic because they could not find a configuration of tables as they did in problem 1. These students did not consider self-inverting elements as the other half did and instead relied on the structure of the tables that they drew. Chase created several configurations of tables for the $\mathcal{Z}_6$ group and ultimately concluded that the two groups are not isomorphic because there is no way to make them "look the same," saying, "we can't get this (a table for the $\mathcal{Z}_6$ group) to look like that (the given table of triangle symmetries)." Another justification two students used was that there were "unequal instances of unique elements" on the main diagonal of $\mathcal{Z}_6$ and "inconsistencies between rows and columns" when comparing the triangle symmetries and $\mathcal{Z}_6$ tables.

All the students (n=11) were successful in concluding that the given groups are not isomorphic. In this problem, we see an almost even split between the number of students who were able to reason from the perspective of isomorphism properties and the students who reverted to their techniques used in problem 1. This suggests that the students who continued to use tables did not yet understand the identity and inverse preserving properties of isomorphism.

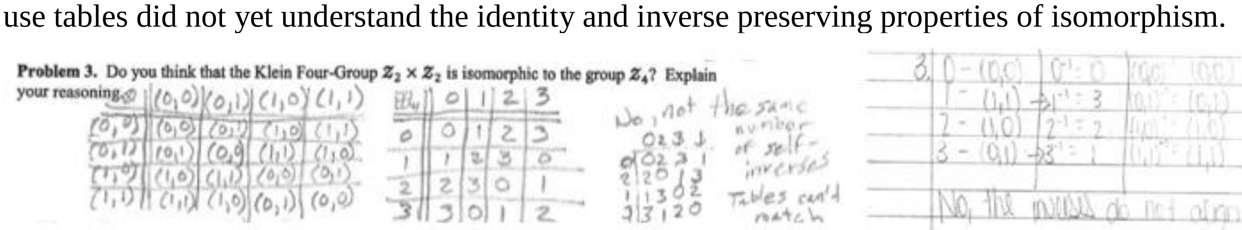


Figure 3. Jimmy's method (left) and Rachel's method (right) for completing problem 3

Problem 3 uses the $\mathcal{Z}_2 \times \mathcal{Z}_2$ and $\mathcal{Z}_6$ groups to reiterate the ideas presented in problem 2. The goal of this problem is to reinforce the students' conceptual understanding gained in problem 2. Upon analysis, data showed that 73% (n=8) of the students reasoned that the given groups were not isomorphic because each had a different number of self-inverting elements. Of these students who recognized the discrepancy in self-inverting elements, four used a table to come to this conclusion and four students did not use a table. In Figure 3, Jimmy draws two different configurations of the $\mathcal{Z}_4$ table and compares them to his written $\mathcal{Z}_2 \times \mathcal{Z}_2$ table before concluding that the groups cannot be isomorphic because they have a different number of "self-inverses." Rachel's work in Figure 3 is an example of the work of students who did not use a table to come to their conclusion but instead created a mapping and found the inverses of each element before concluding that "the inverses do not align."

Alternatively, two of the students did not mention that the groups had a different number of self-inverting or non-self-inverting elements but instead came to the correct conclusion by drawing a table for each group and comparing. One student drew a table and found that the diagonal "contains unequal instants (instances) of unique elements," but did not explicitly state that the two groups possess different numbers of non-self-inverting elements in their work or group discussion. In problem 3, almost three-fourths of the students recognized that the groups had different numbers of self-inverting elements. This suggests that some students were able to transition from table-reliant work to a greater understanding of isomorphism properties between problems 1 and 2.

The goal of problem 4 is to help students grasp the definition of operation preserving functions that has just been presented to them. Preferably, students will use the new definition of operation preserving functions to correctly answer problem 4. Every student was successful in finding a counterexample to show that the given mapping is not an isomorphism. We found that 36% (n=4) of the students did this by reverting to using written tables for the groups and comparing them. These students had more trouble completing the task than their peers who used the definition. One student described that it was hard to use the tables to find "what goes wrong" specifically because there are multiple "wrong" arrangements that make each table appear to not be isomorphic to its counterpart.

Conversely, 64% (n=7) of the students did not use tables to come to the correct conclusion. These students completed the task relatively quickly by finding counterexamples that did not preserve the operations of the groups. Samantha found a counterexample by checking if g preserved the operation of $\mathcal{Z}_6$ when operated on elements $(0,1)$ and $(1,0)$ from the $\mathcal{Z}_3 \times \mathcal{Z}_2$ group. Her work showed
"$g\big((0,1) \otimes (1,0)\big) \neq g(0,1) \boxplus_6 g(1,0) \; \backslash\backslash \; g(1,1) \neq 3 \boxplus_6 2 \; \backslash\backslash \; 1 \neq 5.$"

Altogether, the majority of the students successfully used the new definition of operation preservation to show that the given arrangement was not an isomorphism between the groups $\mathcal{Z}_3 \times \mathcal{Z}_2$ and $\mathcal{Z}_6$. Students who relied on the written tables encountered difficulties using this technique to solve the problem; their reluctance to use the new definition suggests that these students have developed a slightly weaker understanding of isomorphism than their peers.

Finally, homework problem 3 was used to determine if students gained an adequate understanding of isomorphism. Approximately three-fourths (n=8) of the students answered question 3 sufficiently, meaning they showed suitable work to prove that the given function was an isomorphism. Of these students, four explicitly cited the definition of isomorphism and four did not. The four students who did not clearly state the definition of isomorphism showed that the given function was bijective and operation preserving but did not conclude that these factors proved the function was an isomorphism. Since the majority of the students answered homework problem 3 correctly, it suggests a passable understanding of isomorphism. The fact that only half of these students used the definition of isomorphism explicitly in their work could suggests that half of the students do not understand or feel comfortable using the formal definition.

Discussion and Conclusion

Isomorphism is a significant component found in multiple realms of mathematics. Moreover, it is a core concept introduced in beginning Abstract Algebra courses. Previous research shows that, while significant, the concept of isomorphism is "seldom understood by students” (Mena-Lorca & Parraguez, 2016, p. 377), causing the teaching of isomorphism to be a difficult task for Abstract Algebra instructors. To pinpoint and address students’ understandings and misunderstandings of isomorphism, we conducted an in-class study on student responses to an isomorphism investigation that utilizes a radical constructivist approach. The goal of this investigation is to allow students to use previously learned concepts and rudimentary skills to construct their own understanding of isomorphism.

Of the few studies that have been conducted on isomorphism, most analyze students’ reconstruction (Mena-Lorca & Parraguez, 2016) and reinvention (Larsen, 2009, 2013) of theorems on isomorphism. This curriculum deviates from the guided reinvention approach by supporting student construction of the concept of isomorphism. We believe this study expands upon and supports current findings on students’ understanding and provides new insight on student responses within the context of these new curricular materials.

In agreement with past studies, we found that students’ have difficulties reasoning with the concept of isomorphism. This led us to conclude students are not prepared to learn the concept of isomorphism starting with the formal definition, but instead must initially gain an image-based understanding. Students who showed progress in their understanding tended to rely on either written operation tables or individual assignments when finding isomorphic mappings. Student responses to problem 1 of the investigation showed the most consensus and adherence to the instructor’s goal for the problem when compared to student responses to other problems in the investigation. This suggests that problem 1 was the most successful at helping students construct an understanding of isomorphism. Students who relied solely on operation tables in their work throughout the investigation reasoned that for two groups to be isomorphic, their tables must look the same, suggesting that their understanding was purely image-based and supporting the theory that “the context of geometric symmetry can provide a rich and natural context for developing the concepts of group theory” (Larsen, 2009, p. 136). These students informally recognized the properties of isomorphism through conditions for their tables (i.e. the corresponding tables must have an equal number of instances of the identity elements in their diagonals informally requires that the groups have an equal amount of self-inverting elements) but found it difficult to recognize these properties outside of the tables. This suggests that while students’ find the most progress in problem 1, the techniques learned in this problem have the danger of becoming “crutches” throughout the investigation. To attempt to resolve this problem future drafts of the materials could include a smoother transition in the investigation from the table-oriented problems to the formal definition of isomorphism. Contrastingly, students who were able to recognize and consistently use the properties of isomorphism in their work, showed a greater intuition when finding isomorphic mappings. In sum, we have detailed the techniques students use to approach varying challenges while learning the concept of isomorphism. The findings of this study support previous research that suggests students’ understanding of isomorphism is largely reliant on an imaged-based concept of symmetry (in this study, operation tables). Moreover, we found that students who progressed from a strictly imaged-based reasoning to a property-based reasoning demonstrated greater understanding of isomorphism. Even with these findings additional research is needed on how students develop an understanding of isomorphism and the impact of different curricula on students’ understanding.

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