# Influences from Pathways College Algebra on Students' Initial Understanding and Reasoning about Calculus Limits

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*The Pathways to College Algebra curriculum aims to build concepts that cohere with the big ideas in Calculus, and initial results suggest improved readiness for Calculus by students who have taken a Pathways class. However, less is known about how Pathways might influence students' initial understanding and reasoning about calculus concepts. Our study examines similarities and differences in how Pathways and non-Pathways students initially understand and reason about the calculus concept of the limit. Our findings suggest that Pathways students may engage a little more in quantitative reasoning and in higher covariational reasoning, and have more correct and consistent initial understandings. Further, the Pathways students were explicitly aware of how their Pathways class may have benefited their understanding of limits.*

*Keywords*: Pathways College Algebra, Calculus, Limits, Understanding, Reasoning

A critical idea in mathematics education is *coherence* across curriculum (NCTM, 2006; NMAP, 2008; Newmann, Smith, Allensworth, & Bryk, 2001; Schmidt, Wang, & McKnight, 2005). Thompson (2008) argues that coherence should be viewed through *ideas* and *meanings* rather than topical structures and orderings. Such coherence seems to be lacking between calculus and its prerequisite classes, like College Algebra, which often focuses on calculations and procedures (see Blitzer, 2014; Sullivan, 2012). While knowing procedures can help students manually work out calculus problems, it is hard to see how these cohere with the *big ideas* in Calculus of limits, rates of change, and accumulation (see Kaput, 1979; Thompson, 1994).

To address this issue of coherence, a recent curriculum for College Algebra, *Pathways to College Algebra* (Carlson, 2016, hereafter referred to as "Pathways"), aims to build Algebra concepts through quantitative and covariational reasoning. The curriculum was developed specifically to cohere with big ideas in calculus and data has shown that students who used the Pathways curriculum tend to be better prepared to enter Calculus (Carlson, Oehrtman, & Engelke, 2010). However, little work has been done in documenting exactly *what* students who have used the Pathways curriculum do differently than their non-Pathways peers. This study examines one specific area, namely how a Pathways experience might influence students' understanding and reasoning about limits at the *beginning* stage of limit instruction. Our guiding research question is: What differences or similarities are there between calculus students who took non-Pathways algebra versus Pathways algebra, in terms of how they *initially* understand and reason about limits?

## Brief Background on Pathways Curriculum

The Pathways curriculum (Carlson, 2016; Carlson, Oehrtman, & Moore, 2017) was developed to provide a coherent and meaningful course for students that would help them understand the foundational aspects of calculus. The Pathways curriculum was informed by research on learning functions (Carlson, 1995, 1998), the processes of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), mathematical discourse (Clark, Moore, & Carlson, 2008), and problem-solving (Carlson & Bloom, 2005). The curriculum contains modules based on research of student learning and conceptual analysis of the cognitive activities conjectured to be necessary to understand and apply the module's central ideas. Specific concepts that are targeted include rate of change; proportionality; functions: linear, exponential, logarithmic, polynomial, rational, and trigonometric; polar coordinates; vectors; and sequences and series. The curriculum also supports a problem solving approach to mathematics, where students are expected to engage in novel contexts and reasoning to construct mathematics.

## Initial Understanding and Reasoning about Limits

In this section, we articulate our perspective on "initial understanding and reasoning about limits," based on the research literature. Of course, our perspective outlined here will not contain everything that may be involved in understanding or reasoning about limits, because of the fact that our study only deals with understanding at the *beginning* stages of learning limits.

We define "initial understanding" as a student's early *concept image* of limit at the initial stage (Tall & Vinner, 1981), which we would expect to be fairly narrow and incomplete. Also, because some misconceptions are nearly "unavoidable" (Davis & Vinner, 1986) and take considerable exposure to examples, counterexamples, and contexts to address (Cornu, 1991; Przenioslo, 2004; Swinyard, 2011), we are less interested in documenting misconceptions students have. Rather, we are interested in comparing students' initial understanding with standard *informal* definitions of limit. In our study, we examine cases of both the limit at a point, $\lim_{x \to a} f(x) = L$, and the limit at infinity, $\lim_{x \to \infty} f(x) = L$. While the students in our study had not yet discussed limits at infinity in their classes, we wanted to know how they might attempt to understand and reason about them with only the first day of limit instruction. Our informal definition of limit at a point is that the limit of $f(x)$ is $L$ "if we can make the values of $f(x)$ arbitrarily close to $L$… by restricting $x$ to be sufficiently close to $a$… but not equal to $a$" (Stewart, 2015, p. 83). Our informal definition of limit at infinity is that "the values of $f(x)$ can be made arbitrarily close to $L$ by requiring $x$ to be sufficiently large" (Stewart, 2015, p. 127).

We define "initial reasoning" through two aspects of reasoning that the literature has claimed are important for limits. First, Kaput (1979) has stated that "virtually all of basic calculus (the study of change) achieves its primary meaning through an absolutely essential collection of motion metaphors" (p. 289). As such, changing quantities are a part of early reasoning. However, standard curricula often focus heavily on algorithms for finding limits (e.g., Stewart, 2015; Thomas, Weir, & Hass, 2014). Nagle (2013) claims that this approach likely leads students to have "independent, unconnected conceptions" of limits that are based on quantities and computation (p. 3). Consequently, the way students use quantitative reasoning versus computational reasoning is one part of their "initial reasoning about limits."

Second, some researchers have noted strong relationships between covariational reasoning and understanding limits, due to a limit inherently dealing with two changing quantities (Carlson et al., 2001; Carlson et al., 2002; Nagle, Tracy, Adams, & Scutella, 2017). The informal, "as $x$ approaches $a$, $y$ approaches $L$," strongly suggests covariation between $x$ and $y$. Carlson et al. (2002) even claimed that, "Students' difficulties in learning the limit concept have been linked to impoverished covariational reasoning abilities" (p. 356). Because of the importance of covariational reasoning, even at the early stage of learning limits, we consider how students use covariational reasoning as the other part of their "initial reasoning about limits."

## Methods

Twelve Calculus 1 students at a large private university participated in the study. All students had taken College Algebra at the university during the previous year, with five having taken Pathways (P) and the other seven having taking non-Pathways Algebra courses (N-P). The students' Algebra grades and Calculus pre-test scores were similar across the two groups, though three of the N-P students had completed Calculus previously. Students were interviewed about limits the day of or the day after their initial lesson on limits in Calculus 1. Unfortunately, one P student had not attended his calculus class the day limits were introduced, so we excluded him from the study. Because we ended up with only four P students in the data, we are careful to state that the results of this study can only be suggestive, not conclusive. We label the P students as PA, PB, PC, and PD and the N-P students as N-PA, N-PB, N-PC, N-PD, N-PE, N-PF, and N-PG.

The interview contained four questions: (1) Explain the meaning of $\lim_{x \to a} f(x) = L$. (2) If you found the limit, $\lim_{x \to \infty} 4x^2/(x^2 - 5x + 6)$ , what would you be finding? (3) Select the graph(s) [among six graphs given to the students] that correspond to each limit expression, (a) $\lim_{x \to \infty} f(x) = 1$, (b) $\lim_{x \to 0^-} f(x) = -\infty$, (c) $\lim_{x \to 3^-} f(x) = 0$. (4) The equation $v_{orbit} = \sqrt{GM/r}$ relates a satellite's required velocity for a stable orbit, $v_{orbit}$, with its distance from Earth, $r$ (where $M$ is the Earth's mass and $G$ is a constant). What is $\lim_{r \to \infty} v_{orbit}$? After questions 2 and 4, students were also asked to identify any connections they saw from their college algebra class that might have helped them understand limits.

We analyzed the students' responses according to their reasoning and understanding as follows. We first analyzed how students used quantitative reasoning versus procedural reasoning in their responses. Quantitative reasoning was operationalized as using number sense and relationships between quantities to discuss the limits. Procedural reasoning was operationalized as using an algorithm or memorized set of steps to solve the problem, without explaining why the process worked, regardless of whether the student used the procedure correctly or not. However, if the student explained why the process worked, it was coded as quantitative reasoning, rather than procedural. We note that we applied this analysis only to the questions where students *could* potentially compute the limit, questions 2 and 4.

Second, we analyzed students' covariational reasoning behavior from all four questions by classifying individual responses according to the reasoning levels in Thompson & Carlson's (2017) framework: *no coordination*, *precoordination*, *gross coordination*, *coordination of values, chunky continuous covariation*, and *smooth continuous covariation*. We further grouped these levels into "high," "mid," and "low" categories. High covariational reasoning included *coordination of values, chunky continuous covariation,* and *smooth continuous covariation*, because these were less commonly exhibited types of reasoning. Mid covariational reasoning included only *gross coordination*, because it was the most commonly used reasoning level. However, during analysis two subcategories emerged within the mid level: (1) We attended to whether students were *specific* about the quantities involved in the covariation, or whether they used *imprecise* language to refer to the quantities (Leatham, Peterson, Merrill, Van Zoest, & Stockero, 2016). (2) Some students were close to the boundary between *gross coordination* and *coordination of values*, by explicitly attending to the limiting values that $x$ and $y$ were approaching. We labeled this as a new reasoning level, *gross with limiting values (GLV),* as opposed to regular *gross coordination (GC)*, and we consider *GLV* to be on the higher end of the mid category. Finally, low covariational reasoning included *precoordination* and *no coordination*. Also, when students were consistently incorrect about the relationship between *x* and *y* (e.g., interpreting *x* →infinity as *y*→infinite), those instances of reasoning were also coded into the *low* category.

The last step of analysis was to infer student's initial understanding of limits by documenting their description of what a limit was by the end of questions 1, 2, and 3. We decided not to include question 4 here because the students generally struggled with it. We recorded whether the students' descriptions were mathematically correct according to our informal definitions. We also noted whether a student's descriptions were consistent across questions, including for *limit at a point* at the end of questions 1 and 3 and for *limit at infinity* at the end of questions 2 and 3.

## Results

### Procedural versus Quantitative Reasoning

We grouped the students into three categories based on their reasoning: reliance on quantitative reasoning, reliance on procedural reasoning, or reliance on combined reasoning. To illustrate an example of a student who relied on quantitative reasoning, consider N-PA's explanation for how he found the horizontal asymptote in question 2:

> *N-PA*: So as x gets increasingly large, only the most powerful exponents of x are actually going to have much of a difference. ... And so as you get bigger and bigger to 100 or 1,000, or 100,000, then these values here on the bottom become pretty much obsolete or irrelevant. At that point, you can just look at the highest exponent of x [circles $4x^2$over $x^2$]. In those really large number areas, we have two exponents that are equal to each other… so we know that in the end, it's approaching a positive value of 4 at some very, very far distance down the road.

Compare this example of quantitative reasoning in identifying the horizontal asymptote to an example of a student who mostly relied on procedural reasoning. When N-PE was initially asked question 2, he stated, "I don't have the slightest." He continued,

> *N-PE*: I know that I would search for asymptotes. That would be one of the first things that I would search for. … I would look for vertical asymptotes, which is where x is equal to 0 [points to denominator]. I would break that apart which would be -3 and -2, right? Yeah, -3 and -2 [writes $(x-3)(x-2)$ and points to numerator] and that doesn't break up into $x=-3$ or $x=-2$, so there would be vertical asymptotes when $x=3$ and $x=2$. That's where I would start.

This student, upon seeing a rational function, appealed directly to the procedure for finding vertical asymptotes, which is unproductive in this context. The point is that N-PE relied on trying to identify and use a procedure when encountering an unfamiliar question.

We identified a third category of combined reasoning, that we defined as students who used both quantitative and procedural reasoning during these questions. As an example, in Question 4, PD began by using quantitative reasoning to explain how the function $v_{orbit}=\sqrt{GM/r}$ behaves until he became stuck on an inability to remember a specific set of rules.

> *PD*: The way I'm thinking, as *r* is getting bigger, this fraction inside ... is getting smaller inside of the square root. And the denominator is going to continuously get larger ... then the fraction will get smaller. The fraction [pause]. I'm really not sure with this one. The problem is that I can't remember, because usually with square roots and things like that there are all of these rules. … I can't remember what would happen if you would take the square root of that. I mean the square root would give you a larger number or if it gets

even smaller. I can't remember… So I really don't know.

Three of the four P students and one of the seven N-P students relied on quantitative reasoning to answer questions 2 and 4. Three N-P students relied on procedural reasoning. One P student and three N-P students reasoned with combined reasoning. These results suggest a skew for the P students toward quantitative reasoning and a skew for the N-P students toward procedural reasoning. We also note that the three N-P students who relied on procedural reasoning were consistently unsuccessful in completing question 2 and 4.

### Covariational Reasoning

In this subsection, we provide examples of high, mid, and low covariational reasoning, and explain the trends between the P and N-P groups in terms of their covariational reasoning.

**High covariation.** Of all instances of covariational reasoning among the four P students, 12% of their instances were coded in the high category, with three of the four students having instances in this category. Of all instances of reasoning among all seven N-P students, 3% were coded in the high category, with three of the seven students having instances in this category. To exemplify reasoning at these higher levels, consider PA, who displayed continuous covariational reasoning in his response to the expression $\lim_{x \to 3+} f(x) = 0$ in question 3. When asked to explain why he only looked at x-values coming from the right side, he said,

> *PA*: The little positive symbol right there, by 3. It's asking for values that are just bigger than three… In graph number one [points to a graph], there is a hole and so at the value of 3 there is no output for $f(x)$. But if we were to get infinitely close to three, with values just bigger than three like 3.1, 3.01, 3.001. We're getting closer to the output value of zero.

**Mid covariation.** Students most commonly reasoned at this level of covariation. Of all reasoning instances for the four P students, 85% were in the mid category, and of the seven N-P students, 69% of all reasoning instances were in the mid category.

To illustrate the differences between *specific* versus *unspecific* reasoning, and *GLV* versus *GC* reasoning, consider the following examples. First, when N-PD was justifying his choice of graph for $\lim_{x \to 3+} f(x) = 0$ in question 3, he stated, "As *it* moves from the positive side, *it* looks like *it* will be 3" [emphasis added]. Note that N-PD used the ambiguous language "it." Further, this statement suggests basic *GC* because of the generic description of "increasing." Thus, we consider this reasoning instance to be *unspecific* and at the level *GC*.

By contrast, consider N-PD's response to the same question:

> *N-PD*: As $x$ approaches 0 from the negative side. We have $f(x)$ approaches negative infinity. So, this graph [point to a graph] is approaching zero from the negative side [motions horizontally across the left side of the $x$-axis]. And as it does the value of $f(x)$ plummets to negative infinity [motions vertically along the bottom half of the $y$-axis].

Unlike N-PD, PC always specified which quantity he was attending to, whether $x$ or $f(x)$. Also, in addition to general statements about "increasing," PC was specific about the values the quantities $x$ and $f(x)$ were approaching, zero and infinity. Thus, we consider this reasoning instance to be *specific* and to be at the level *GLV*.

Generally, the P students were specific in their reasoning more often than N-P students. P students were specific for 72% of all reasoning instances in the mid category, while N-P students were specific for 38% of all reasoning instances in the mid category. However, within just the mid category, the P and N-P students had similar percentages of reasoning instances at the *GC* level versus the *GLV* level, with about two-thirds of all mid category reasoning instances being at *GLV*. Table 1 summarizes the results for the mid category of covariational reasoning.

*Table 1. Results for specific versus unspecific and GLV versus GC within the mid category*

| | Specific versus Unspecific | | GLV versus GC | |
|---|---|---|---|---|
| | Specific | Unspecific | GLV | GC |
| P students | 72% | 28% | 65% | 35% |
| N-P students | 38% | 62% | 67% | 33% |

**Low covariation.** Of all reasoning instances among the P students, 3% were in the low category, and among all instances for the N-P students, 28% were in the low category. A common example of this category was when students reasoned about a graph as an object, rather than as two variables coordinated together. For example, consider N-PF's reason why she believed that $\lim_{x \to 0-} f(x) = -\infty$ should be matched with a graph of the horizontal line *y* = 1 with a hole at *x* = 0:

> *N-PF*: So we have 0 to the negative, which means we're going to look at a point just smaller than 0… And you see that it's going to approach negative infinity. It is going to go on forever in the negatives [gestures horizontally along the x-axis towards negative infinity].

It appeared she was thinking of $0^-$ and $-\infty$ both in terms of *x* alone, without attention to *y*, meaning she was not coordinating two variables. Another typical reasoning among the low category instances, typified by N-PF for the same limit and graph, was, "This [graph] makes somewhat sense because the line goes on forever and it's at 1."

Another type of reasoning we categorized as low was when students attempted to covariate but mixed up *x* and *y* values. As an example, N-PG did so for the limit in question 2 when she said the limit would be approaching a vertical asymptote, imagining that *y* was approaching infinity, rather than *x*. She showed this same confusion in question 3 when she incorrectly claimed that the limit $\lim_{x \to \infty} f(x) = 1$ indicated a vertical asymptote at *x* = 1.

In summary, the results for the students' covariational reasoning suggest a similarity between the two groups in that both had a majority of reasoning instances in the mid category. However, we can see that P students' covariational reasoning was overall skewed somewhat higher than the N-P students, and that they were more specific in articulating that covariation.

**Initial Understandings for Limit**

Finally, no student had a perfect initial understanding for limits, as we expected. However, five of the students across the two groups had understandings that were "correct," according to our informal definitions, and whose descriptions of a limit remained consistent across the interview questions. As an example, by the end of question 1 PC had described a limit as, "We're not necessarily looking for the value of the function at a specific *x* point, but what *f*(*x*) is approaching at that certain point, from both sides." This description remained consistent through question 3 as well. Three of the four P students were in this group, as well as two of the seven N-P students. Note that both of the N-P students in this group had completed calculus before, while none of the P students had. We find it impressive that these three P students had correct, consistent limit definitions after just one day of learning about them.

Three of the N-P students and one of the P students often referred to the limit as giving the slope of $f(x)$, possibly based on how their instructors introduced limits during class. The other two N-P students simply had many varying ideas throughout the interview or would mix up the input and output values when the limit approached infinity.

### Connections to Algebra Experience

When asked for connections to their College Algebra course during the interview, students from both groups mentioned several topics they remembered. However, four of the seven N-P students only listed computational procedures, such as finding vertical asymptotes or finding the inverse of a function. The other three N-P students primarily cited graphing as a connection. The N-P students generally were explicit in stating that their College Algebra course was not helpful in learning about limits in calculus.

By contrast, three of the four P students discussed a change of thinking. Students said that their College Algebra courses helped them reason on their own or gave them an understanding of how concepts work. The remaining P student mentioned both this change in thinking and procedures like discontinuities and parent graph functions. All of the P students indicated that their College Algebra experience directly helped them in learning limits in calculus.

## Discussion

In discussing the trends seen in the results, we again caution that our small sample is only suggestive, and cannot imply generalization to the larger Calculus student population. However, within our small sample, we certainly observed differences in trends for how the overall group of P students reasoned about and understood limits compared to their N-P counterparts. Of course, there was overlap in how the students reasoned about limits. For example, students from both groups were seen to reason quantitatively and procedurally. Students from both groups were seen to reason at lower and higher levels of covariational reasoning. Yet, taken in aggregate, P students were shifted more toward using quantitative reasoning than procedural reasoning, and were overall shifted somewhat toward higher levels of covariational reasoning. Their initial understandings for limits were also more on the correct/consistent side of the spectrum. This certainly does not mean that N-P students cannot engage in these types of reasoning nor hold those types of understandings for limit, as seen in our results. But it does suggest a small net effect for the students having taken Pathways College Algebra in using higher reasoning and having better developed personal meanings. In other words, the Pathways curriculum seems to *cohere* with initial limit instruction, which is an important aspect of sound curriculum (NCTM, 2006; NMAP, 2008; Newmann et al., 2001; Schmidt et al., 2005; Thompson, 2008). This coherence suggests a possible advantage for P students when encountering the difficult limit concept for the first time. While it may only be small, if it is combined with a net advantage for other concepts as well, such as the derivative and integral, it begins to build a picture as to why P students might be more successful in calculus (see Carlson, Oehrtman, & Engelke, 2010). In fact, the students themselves seemed aware of the ways in which their Pathways curriculum connected to the limit concept they were in the process of learning about.

We suggest building on this work by sampling a larger group of Pathways versus non-Pathways students to see if the trends observed in our small sample hold for that larger group. Our study suggests that there may be differences, and such future work would be needed to gain the desired generalizability to the larger student population.

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