Abstract Algebra Students' Function-Related Understanding and Activity

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Functions play a fundamental role both in abstract algebra and earlier courses in the mathematics curriculum. Yet little attention has been paid to how students' understanding of function (informed by their prior experiences) supports or constrains their activity when dealing with functions in abstract algebra. In this study, we report on six abstract algebra students' understanding of function, their function-activity in abstract algebra tasks, and the degree to which their understanding of function from prior experiences is connected to their understanding in this new setting. We conclude with two cases contrasting the activity of two students with divergent levels of connection between their function understanding and the abstract algebra setting. In general, we found that function served an important role in students' activity and provides implications for instruction and future research.

Keywords: Abstract Algebra, Functions, Student Understanding

Functions are one of the core topics threaded throughout the mathematics curriculum. In abstract algebra, students encounter a number of important classes of functions including isomorphisms and homomorphisms. The treatment of functions in this setting is often more formal and abstract; however, students' extensive exposure to functions in prior courses likely plays a role as they grapple with new function contexts and definitions. The degree to which this occurs is particularly pertinent due to the extensive documentation of complexities involved with understanding functions at the secondary level (Oehrtman, Carlson, & Thompson, 2008). Understanding functions involves integrating function-properties (e.g., Slavit, 1997), flexibly understanding multiple representations (e.g., Schwarz, Dreyfus, & Bruckheimer, 1990), leveraging appropriate metaphors (e.g., Zandieh, Ellis, and Rasmussen, 2017), and moving beyond action conceptions to process and object conceptions (e.g., Breidenbach, Dubinsky, Hawks, and Nichols, 1992). In parallel, the abstract algebra literature illustrates that students often struggle with aspects of specific functions such as isomorphisms (e.g., Leron, Hazzan, & Zazkis, 1995), binary operation (Melhuish & Hicks, 2018), and homomorphisms (e.g., Rupnow, 2017). With these results in mind, we developed a survey and interview study to address:

1. What are students concept images of functions at the end of an abstract algebra course?
2. How do they see functions from prior courses as connected to functions in abstract algebra?
3. How does their understanding of functions play out in their abstract algebra activity?

Literature Review

The complexities involved in understanding function have been well documented. Students have been found to possess several alternate or incomplete conceptions of function that can persist even throughout the secondary and undergraduate level (Oehrtman et al., 2008). For example, students may interpret functions as necessarily having an explicit symbolic rule (e.g., Vinner & Dreyfus, 1989; Thompson, 1994). Students may also struggle with definitional properties such as delineating between the requirement for a well-defined function and that of a function being injective (Dubinsky & Wilson, 2013). Further, their conceptions of functions may reflect different degrees of sophistication such as in Breidenbach, Dubinsky, Hawks, and Nichols' (1992) documentation of students conceiving of functions as actions, process, or objects.

Understanding of function has been treated through different lenses including the aforementioned action, process, and object hierarchy. Slavit (1997) posited an alternate route of function understanding relying on important properties of functions and distinguishing between functions possessing or lacking properties. Another marker of understanding of function is proficiency with multiple representations of functions. Numerous researchers have documented students' preferences for a particular representation even when alternate representations would be supportive (e.g., Knuth, 2000), students' lack of flexibility moving across representations (e.g., Akkoç, & Tall, 2002), and even students seeing alternate representations as unique functions (Elia, Panaoura, Eracleous, & Gagatsis, 2007). As is the case with representations, students may also leverage multiple function metaphors while reasoning about functions. Such metaphors may reflect the input-output machine (Tall, McGowen, & DeMarois, 2000) or directionality between sets (e.g., Lakoff & Núñez, 2000). Zandieh et al. (2017) identified five clusters of metaphorical expressions with which students engaged in linear algebra: input/output, traveling, morphing, mapping, and machine. Properties, representations, and metaphors provide additional components to be situated in a students' larger concept image of function.

The concept of function then plays a vital role in more advanced courses such as abstract algebra. While little research has treated function explicitly at this level, existing literature in abstract algebra suggests that students struggle to develop rich conceptions of abstract algebra concepts that rely on functions (Dubinsky, Dautermann, Leron & Zazkis, 1994; Hazzan, 1999). Students in abstract algebra tend to struggle with particular kinds of functions such as isomorphisms and binary operation. For example, Leron et al. (1995) found that students struggled with constructing specific isomorphisms and formulating definitions about isomorphisms. Rupnow (2017) shared cases where students struggled with homomorphism when they did not have metaphor flexibility. Melhuish and Hicks (2018) documented that students may bring some of the same function representational limitations to the context of binary operations. In sum, the results from prior research suggest that explicitly studying student conceptions of function may provide insight into their abstract algebra activity.

Theoretical Orientation and Analytic Framework

In this paper, we rely on two key constructs to make sense of students' understanding: Tall and Vinner's (1981) *concept image* and Zandieh et al's (2016) *unified notion of function.* A student's understanding of function involves not only the words used to specify the concept (personal concept definition), but also all of the surrounding cognitive structures (concept image). These various components may or may not be coherent and they may or may not align with mathematics' communities accepted definition for a given concept. In terms of functions, a number of components have been associated with concept images including metaphors (e.g., Zandieh, et al., 2016), representations (e.g., Hitt, 1998), properties (e.g., Tall & Vinner, 1981), and evoked examples (reflecting a students' *personal example space*, Sinclair, Watson, & Mason, 2011). Due to space limitations, we share the specific categories from our analytic framework in Table 1.

Our primary research goal was to address each of these components of abstract algebra students' concept image of functions at the completion of an introductory course. Further, we use Zandieh et al's *unified notion of function* to address the degree to which students understood "various constructs [of functions] as examples of the same phenomenon" (p. 24). That is, did students see functions presented in abstract algebra as instances of their larger function concept? We address this question through analysis of both students' self-reported understanding and their activity as they engaged with relevant tasks. We conjectured the connectedness of their function understanding would play out through explicit questions about functions in abstract algebra, explicit reference to functions in their abstract algebra activity, and components of their function concept image implicitly playing out in their activity.

Methods

Data Collection

Surveys were given to four undergraduate-level modern algebra classes at two public universities. The survey was composed of one part concerning functions in general and another part concerning homomorphisms and kernels in group theory. In the first part of the survey, students were prompted to provide formal and informal definitions of function, examples of functions, and representations for functions. In the second part, students provided formal and informal definitions of group homomorphism, and kernel. They also were given a series of tasks where they needed to leverage the definition of homomorphism or kernel to address prompts in particular contexts (such as determining if a given map is a homomorphism.) In addition to the surveys, we conducted six semi-structured follow-up interviews (three at each university) with the goal of obtaining a more robust interpretation of the participants' survey responses. The interviews included additional tasks that the students were asked to complete including addressing homomorphisms in the context of Cayley Tables and function diagrams, and producing formal proofs of standard homomorphism and isomorphism prompts. Two such prompts include determining if the function diagram in Figure 1 could potentially be of a homomorphism, and identifying the kernel for the homomorphism in the following map from $\mathbb{Z}$ to $\{i,-i,1,1\}$:

$$\Phi(n)=i^n.$$

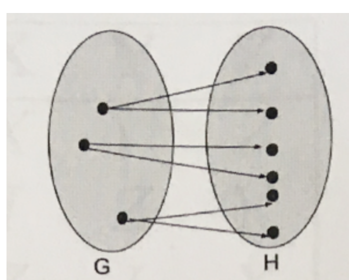

Figure 1. A Diagram Representing a Non-Function

At the end of the interview, the participants were prompted to reflect on functions in their abstract algebra class by identify if and what functions were in the subject. They were then asked to reflect on whether "functions in modern algebra the same as functions from high school?"

Analysis

To analyze the transcripts of the interviews, each of the four authors independently open coded (Strauss & Corbin, 1990) the transcripts looking across all prompts. Through this process, a coding framework was developed to target specific aspects of student thinking that were deemed pertinent to the research questions. In particular, this framework included the properties that students attended to, the metaphors (adapted from Zandieh et al. 2017) and representations (adapted from Melhuish 2015) and Mesa 2004) utilized, the students' evoked example space for functions and non-functions, and the similarities and differences that the students noted between functions in abstract algebra and functions in lower level courses. The four authors then independently coded the transcripts (for all items related specifically to functions as a general construct) using the framework. From these coded transcripts, profiles for of the six cases were compiled leveraging the five targeted categories.

We then returned to the transcripts to further unpack the activity on the second set of prompts: prompts where students engaged in representations and proofs related to homomorphism and isomorphism. These transcript portions were analyzed with the intent of exploring whether a student's conceptions of functions aligned with their abstract algebra activity (implicitly or explicitly), and the degree to which their function conceptions appeared to support or constrain their abstract algebra activity.

Results

In Table 1, we share the variety of ways functions were addressed by our participants in terms of their definitional properties, metaphors, representations, and evoked examples. These components stem from analysis of the general function prompts.

	Properties	Metaphors	Representations	Examples	Functions in AA
Student A	WD	T, Mp	S, G, V, E	F, AA	Different Domains, New Properties
Student B	WD, ED	Mp	S, G, V	F, AA	Different Reps, Expansion
Student C	WD, ED	IO, T, Mp	S, G, D	F, AA	Same, Restructured
Student D	WD, 1-1	IO, Mh, Mc, Mp	S, G	F	Different Reps, More Complex
Student E	NA	T, Mp, Mc	S, V	F	Expansion
Student F	WD	T, Mp	S, G, V	F	Expansion

Properties: WD: well-defined, ED: everywhere-defined, RB: relation-based, EB: equation/rule-based;
Metaphors (adapted from Zandieh et al., 2016): T: traveling, IO: input/output, Mp: mapping, Mc: Machine, Mh: Morphing
Representations (adapted from Mesa, 2004; Melhuish, 2015): S: Symbolic Rule, G: Graphical, V:Verbal, E:element-wise defined, D:Diagram
Examples: F: familiar secondary level algebra functions, AA: abstract algebra context

Table 1. Evoked Components of Abstract Algebra Students' Concept Image of Function

Across our six cases, we note significant differences in the students' evoked concept images. In terms of definitional properties, five of the six students articulated some understanding of well-definedness. However, in one case, the student had not delineated well-defined from a map being one-to-one. In a second case, the student relied on a function as a rule with no additional required properties. In terms of metaphors, mapping was leveraged by all of our students. Various individuals leveraged it to greater or lesser success depending on a number of other factors. In terms of representations, all students had a dominant image with explicit symbolic rules. This is not surprising in light of both the literature and the common usage of such representations (as in Melhuish's (2015) curriculum analysis.) Student D, E, and F particularly leaned on symbolic rules. In terms of examples, we saw a similar trend where Student D, E, and F shared examples of typical (explicit, symbolic rule) functions from earlier settings such as $f(x)=x^2$. In contrast, Student A, B, and C all provided examples that were particular to abstract algebra such as functions whose domain was dihedral group.

We then analyzed how students were seeing functions in abstract algebra as the same or different from high school. In general, the students reported that functions in abstract algebra expanded ideas from functions including new qualities such as properties or representation types. However, we note that even though students made these statements, half of the students did not provide examples in an abstract algebra context specifically even with explicit prompting (after having engaged with abstract algebra function prompts) leading us to question the depth of this declared unified conception. To further instantiate the trends in our data, we share two contrasting cases: Student D and Student C.

Case 1: Student D

Throughout the interview, Student D used typical functions from the secondary algebra and calculus settings when prompted to provide examples of functions and struggled to provide specific examples of functions in an abstract algebra context. Moreover, when asked if she viewed functions in abstract algebra as the same as functions in previous courses, she stated that they are "completely different" and explained:

> That's what threw me off from the very beginning, was the functions weren't the same. It was a totally different way of thinking. I mean, you're not thinking in terms of ... I'm thinking in groups.

She explained that she had previously relied on graphical representations of functions to aid her understanding and that the lack of graphs to represent functions in abstract algebra presented challenges for her understanding. Student D suggested that functions in abstract algebra were too large to draw pictures, as they could involve sets such as the set of integers. Overall, throughout the interview Student D's responses did not suggest that she connected her understanding of functions in previous courses to this context.

In the second half of the interview regarding the concepts of homomorphisms and kernels, Student D's disconnect between functions in prior settings and functions in abstract algebra is made particularly clear. When presented with Figure 1 and asked if the function diagram could represent a homomorphism, Student D responded that "I would say [...] is a homomorphism because all of the elements in G get mapped to a particular H value," Thus, Student D is attending to the need for every element in the group G to be mapped to some element in H.

However, when explicitly asked if the diagram represented a function, Student D correctly identified that this diagram fails to meet the requirements of a function: "two x values with different y values in it wouldn't be able to be defined as a function." Meanwhile this does not perturb her previous classification of this diagram as a possible homomorphism. We interpret this as further evidence of Student D's disassociation of the concepts of function and homomorphism.

This disassociation continues to play out in the portion of the interview regarding kernels. While Student D's definition of a kernel of a group homomorphism, "the set of elements in group G that mapped to the group H, to the identity element", is largely correct; she continued:, "One group, one set of elements is going to map to another set of elements, but, in a sense, the reversal map from that final group to the initial group is what the kernel is, so it maps." Thus, we see that Student D does not necessarily see the kernel of a group homomorphism as a pre-image, but rather the image of an inverse function.

Her response to the Kernel Task provides further evidence that she may be working with an action conception of the homomorphism. When asked for the kernel she explained, "... I wrote that the kernel was zero[1], because you would get one, which was your identity element in H." Student D's kernel candidates focused on identifying a single element of the set of integers which maps to the identity in H. She was testing individual values in the function, but not considering the full preimage of the identity. Such focus on individual pairs of input/outputs likely reflects an action conception of this mapping. If a student lacks a process understanding of function, they may be limited to proceduralized ways of dealing with inverse and preimage (Oehrtman, et al., 2008). When explicitly asked if kernels can have multiple elements, Student D agreed. When further probed about this particular map, she identified one more element, but remained focused on individual inputs and outputs.

Student D presents a case of a student who did not appear to have robust connections between her prior function knowledge and the abstract algebra setting. Further, we may reasonably conjecture that her limited conceptions of function implicitly constrained her ability to work with the kernel concept, a concept that necessitates ability to deal robustly with preimage.

Case 2: Student C

In contrast to Student D, Student C flexibly leveraged function metaphors, attended to important properties of functions, and provided an array of examples and representations of functions. Notably, Student C provided examples of functions situated in the abstract algebra context throughout the interview pairing standard secondary algebra examples (e.g., $f(x) = x^2$) with abstract algebra examples ($h(a,b)=b^2$) when prompted to share examples of functions. This integration was further evidenced when Student C was asked explicitly to address functions in abstract algebra listing out typical functions in this setting including isomorphisms, and homomorphisms. When asked if functions seen in their abstract algebra class are the same as functions that they've seen in other classes, Student C explained:

> But I mean, when we go through the isomorphisms and the homomorphisms, we're really going back to those simple kind of equations that we did in the beginning of algebra. Or in linear algebra kind of thing. It's not necessarily like we're coming up with whole new ideas. It's just restructuring them.

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[1] Note: This student also suggested another singleton candidate for the kernel as she worked to make sense of the identity in H. Unpacking this portion of her response is beyond the scope of this paper.

[2] Note: The student is treating the input element as an ordered pair, but only included the single set of parentheses in their notation.

In contrast to several of the other participants in our interviews, Student C treated functions in abstract algebra as naturally connected to functions from other courses.

We also saw this play out in Student C's engagement with abstract algebra specific tasks. She responded to the diagram in Figure 1 by immediately evaluating if the diagram was of a function.

> The second one is kind of what I was talking about earlier with function that everything has to be taken to exactly one spot. I feel like reverse it would've been fine. Like it was taking *H* to *G*. I'm trying to think of a function that would do this and really there's not one because it's not a function.

She concluded the map could not be a homomorphism because it is not even a function. This consideration to a homomorphism being a function evidenced her connected knowledge.

A second case where we witnessed Student C's connected function knowledge was addressing kernels. Student C explained the kernel as, "…the group of elements, like if you have a homomorphism, let's call it Φ from *G* to *H*. It's the group of elements in *G* that get mapped to the [identity] element of H." She leveraged mapping metaphors for function and was easily able to approach the preimage of a function without constructing a map from the codomain to the domain. For example, when identifying the kernel from the Kernel Task, Student C explained:

> So I said that the kernel of phi was actually the integers times 4. So with that, it was because we didn't have some element ... I just called it the first group *G*. Well, $\mathbb{Z}$. So some element of that takes *i* to that power and gives us out 1, which would be the identity element for *H*. Because is gonna i^1give us *i*. i^2 is gonna give us - *i*, so on. So in order for *i* to be taken to a certain power and give us 1, it needs to be a multiple of 4. And that's to do with ^2i is negative 1. So negative 1 times negative 1. And as long as that's a multiple of 4, we're good to go.

These instances were representative of the way that Student C engaged in tasks. Her function understanding appeared to play a supportive role in her abstract algebra activity.

Discussion

This report addresses six undergraduate abstract algebra students' understanding of functions and provides two cases to illustrate how these understandings play out in their abstract algebra activity. As this work is exploratory, we are not attempting to make generality claims. Rather these varied cases provide images of the complex ways that function knowledge plays out in abstract algebra. The students in this study ranged in terms of their evoked concept image of functions. While all students' concept images contained explicit symbolic rules, several students saw the rule as essential for functions. These same students tended to evoke examples of typical function families such as polynomials without connections to abstract algebra. In contrast, the other students had more varied representations and evoked examples including those explicitly connected to abstract algebra. As seen with Student D, this unified understanding may play a supportive role in abstract algebra activity.

This work has several implications. First, we see that even students in an advanced mathematics course towards the end of their undergraduate tenure can struggle to grasp the complex and nuanced concept of function. From a research standpoint, we may want to explicitly consider the role of function understanding in student activity in advanced mathematics. From a teaching perspective, instructors may want to take stock of students' function understanding even in advanced courses. Second, instructors may want to attend to the ways their students' function understanding plays out in courses and consider how one might more proactively connect their prior function experiences with the new types of functions found in abstract algebra.

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