Multiple Representation Systems in Binomial Identities: An Exploration of Proofs that Explain and Proofs that Only Convince

Elise Lockwood — Oregon State University
John S. Caughman IV — Portland State University
Keith Weber — Rutgers University

Abstract: In the mathematics education literature on proof, there is a longstanding conversation about proofs that only convince versus proofs that explain. In this theoretical report, we aim to extend both of those ideas by exploring proofs in the domain of combinatorics. As an example of an affordance of the combinatorial setting, we explore proofs of binomial identities, which offer novel insights into current distinctions and ideas in the literature about the nature of proof. We demonstrate examples of proofs that can be explanatory or convincing (or both), depending on how a person understands the claim being made (which we refer to as their preferred semantic representation system). We conclude with points of discussion and potential implications.

Keywords: Proof, Proofs that explain and convince, Combinatorics, Binomial identities

Introduction and Motivation

An interesting algebraic question to ponder is why the sum of the binomial coefficients equals 2^n (that is, why $\sum_{k=0}^{n}\binom{n}{k} = 2^n$). For us (and perhaps for others), if we consider actually expanding and summing the left-hand side of the equation, the fact that it simplifies so nicely to the expression 2^n feels a bit like an algebraic miracle. If we ask for a justification of this equation, someone may give a straightforward counting argument, noting that both sides count the same set – namely all possible subsets of all sizes from a set of *n* distinct elements. We may find that counting argument to be convincing and also explanatory in terms of why each expression represents a process that counts the same set of outcomes. Following such a combinatorial argument, we could be *convinced of the truth* of the algebraic relationship without gaining the desired insight into the *algebraic* mystery that we originally observed.

The proof literature has long articulated such a distinction between proofs that only convince and proofs that explain (e.g., Hanna, 1990; Hersh, 1993; Steiner, 1978; Weber, 2010), and it has been pointed out that this distinction is not a simple dichotomy (e.g., Hanna, 2000; Raman, 2003; Stylianides, Sandefur, & Watson, 2016;). Generally, proofs that only convince are characterized as proofs that demonstrate that a proposition is true but without necessarily providing particular insight into *why* it might be true. Proofs that explain are characterized as proofs that do give some indication as to why a particular proposition is true. In this theoretical report, we aim to extend both of those ideas by exploring proofs in the domain of combinatorics. We believe that, generally, the combinatorics can provide an insightful context in which to study questions related to the practice of proof. To demonstrate an affordance of the combinatorial setting, we explore proofs of binomial identities, which offer unique insights that extend useful ideas about the nature of proof. By exploring mathematical examples in a combinatorial setting, we offer examples of proofs that can be explanatory or convincing (or both) depending on how a person understands the claim (which we refer to as a preferred semantic representation system).

Background Literature and Relevant Theoretical Perspectives

How are we taking proof?

Weber and Alcock (2004) say, "When asked to prove a statement, professional mathematicians and logically capable mathematics students all share the same goal – to produce a logically valid argument that concludes with the statement to be proven" (p. 210). We draw on this statement and use a definition of proving as *the process of producing a logically valid argument that concludes with the statement to be proven*. We follow Stylianides, et al. (2016) in distinguishing between proof and proving in the following way: "we consider *proving* to be the activity in search for a proof, whereby proof is the final product of this activity that meets certain criteria" (p. 20). In this paper, we are interested broadly in both proving and proof, and we will clarify if we are exclusively referring to one or the other. In the examples we explore in this paper, the statements to be proven are statements that relate expressions involving binomial coefficients. These expressions are known as binomial identities.

Multiple purposes of proof

The mathematics education literature reports a number of purposes that proof and proving play in the domain of mathematics. One primary reason for proof in mathematics is to convince a reader that a theorem is true. This is typically proposed as a main purpose for proof, especially for research mathematicians. For example, Hersh (1993) notes that "in mathematical research, [proof's] primary role is convincing" (p. 398), and he points out that for the mathematics community, "proof is *convincing argument, as judged by qualified judges*" (p. 389, emphasis in original). Here we interpret that convincing means that one understands the necessity of the conclusion following from the premises, but without the additional constraint that the tools and relationships one wants to see employed are necessarily the only tools and relationships used.

Even though convincing is an important purpose of proof, researchers (e.g., Hanna, 2000; Hersh, 1993; Weber, 2010) are quick to note that simple formal deduction, which may technically prove a theorem, is not why mathematicians value proof and is not what they view as the sole purpose of proof. For instance, Weber (2010) argues that mathematicians value proofs not just because they show that a statement is true, but because they provide additional insight into mathematical content or into the practice of proving. As another example, Hersh (1993) says, "More than whether a conjecture is correct, mathematicians want to know why it is correct. We want to understand the proof, not just be told it exists" (p. 390). These sentiments suggest that proof may be useful for additional reasons than demonstrating the veracity of a theorem.

These multiple purposes of proof highlight a distinction in the literature between proofs as explanatory and proofs as convincing. Hanna (1990) reports that a proof is valued for bringing out essential mathematical relationships rather than for merely demonstrating the correctness of a result. She distinguishes between proofs that *prove* and proofs that *explain*. She points out that a proof that proves "shows only *that* a theorem is true; it provides evidential reasons alone" (p. 9), while a proof that explains "also shows *why* a theorem is true; it provides a set of reasons that derive from the phenomenon itself" (p. 9). A similar dichotomy is also articulated in Hersh (1993), and he distinguishes between proof in a research setting and proof in a classroom setting.

Defining proofs that explain

We now discuss the literature on what it might mean for a proof to explain. There are several ways in which researchers characterize proofs that explain. Hanna (1990) clarifies that she prefers "to use the term *explain* only when the proof reveals and makes use of the mathematical ideas which motivate it," (p. 10). She follows Steiner (1978) by saying that "a proof explains when it shows what "characteristic property" entails the theorem it purports to prove" (Hanna, 1990, p. 10). According to Weber and Alcock, a proof that convinces is "an argument that establishes the mathematical veracity of a statement. Such proofs are typically highly formal, and their function is to remove all doubt that a statement is true" (p. 231). A proof that explains, on the other hand, is "an argument that explains, often at an intuitive level, why a result is true" (p. 231). In another approach, Weber (2010) conceptualizes a proof that explains as one that "allows the reader to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (p. 34). Common to all of these characterizations is the idea that a proof that explains offers some insight into *why* a statement is true (or false). In addition, Stylianides, et al. (2016) refer to literature that defined what it meant for a proof to be explanatory for a prover, "namely, whether the proof illuminated or provided insight to a prover into why a mathematical statement is true (Bell, 1976; de Villiers, 1999; Hanna, 1990; Steiner, 1978) or false (Stylianides, 2009)" (p. 21). Stylianides, et al. (2016) consider proving activity to be "explanatory for the prover (or provers) if the method used in a proof provided a way for the prover to formalize the thinking that preceded and that illuminated or provided insight to the prover into why a statement is true or false" (p. 21).

Stylianides, et al. (2016) is particularly relevant to our work, as they challenged and extended the typical distinction between proofs that convince and explain, especially questioning the assertion that proofs by mathematical induction are necessarily not explanatory. They explore ways in which proofs by mathematical induction may be explanatory for students, and they frame what conditions might best facilitate this phenomenon. We hope similarly to further the conversation about proofs that convince and explain by using proofs of binomial identities.

Our characterization of proofs that explain

We follow Weber (2010) in using Weber and Alcock's (2004) distinction between semantic and syntactic proof production as a way of conceptualizing proofs that explain. Weber and Alcock (2004) identify two qualitatively different ways in which someone might produce a correct proof. They define a *syntactic proof production* as "one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In a syntactic proof production, the prover does not make use of diagrams or other intuitive and non-formal representations of mathematical concepts" (p. 210). In contrast, they define a *semantic proof production* to be "a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). The authors clarify that an instantiation refers "to a systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual […] What is crucial is that the prover use these instantiations in a meaningful way to make sense of the statement to be proven and to suggest formal inferences that could be drawn" (p. 211). We interpret that in semantic proof productions, students meaningfully draw on some instantiation of a mathematical object or idea that may be external from the situation at hand.

Even more specifically, Weber (2010) draws on these ideas of intuition and instantiations to provide a definition of an explanatory proof. He notes that, "often, students and mathematicians will use [semantic] reasoning as a basis for constructing a formal proof" (p. 34). In this way, the informal, meaningful semantic reasoning might guide the development of a formal proof. Weber says, "I conceptualize a proof that explains as a proof that enables the reader of the proof to reverse the connection – that is, this proof allows the reader to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (p.

34). We interpret, then, that a proof that explains allows for a prover to make meaning of whatever formal representation system he or she may be working with in order to connect ideas to some semantic system.[1] We thus follow Weber in using instantiations and the notions of semantic proof production (and comprehension) as we define proof that explains.

Finally, as Weber's (2010) definition suggests, he takes a reader-centered perspective on explanatory proof. Indeed, this approach resonates with us, as what constitutes a meaningful semantic system could vary from person to person, according to the content or robustness of their particular concept image (Tall & Vinner, 1981). We thus follow Weber (2010) who emphasizes that proofs that explain are from the perspective of the reader (or the prover).

Mathematical examples

Semantic representation systems

Weber (2010) discussed the semantic representation systems (SRS), which he attributes to Weber and Alcock (2009). An SRS is the system in which a reader's (or a prover's) semantic reasoning may take place, and we interpret an SRS as a mathematical perspective in which a person is interpreting a claim being made. Ultimately, we argue that to ask whether or not a proof is convincing or explanatory, we ought to consider in which SRS(s) a proof is being produced or comprehended. Broadly, these semantic contexts represent the particular perspective in which a prover is proving (or a reader is comprehending) a proof. Often, the statement that is meant to be proven is expressed in a particular symbol system, which may be interpreted in a number of ways. The main idea we are proposing is that proofs (and proving activity) exist within a particular SRS, each of which represents different ways of interpreting and making meaning of the same (symbolically identical) statement to be proven.

We are using Weber's (2010) notion of SRSs as a way to make sense of a variety of proofs of the binomial theorem, and we use the notion of SRSs to understand two important phenomena related to proofs that only convince and proofs that explain. First, in terms of proofs that explain, we use SRSs as a way to articulate *what is being explained* in a proof that explains. By specifying in which SRS we are working, we can gain clarity about what is being mathematically explained. Second, we use SRSs as a way to consider a mechanism by which a proof may be convincing but not explanatory. Specifically, there may be some translation that occurs between SRSs in order to complete a proof. And if a person is trying to prove a claim in one SRS (SRS1), but then translates to another SRS (SRS2) to prove the statement, the proof may be *convincing* to the prover in SRS1 while it is explanatory in SRS2. Thus, a proof may be convincing (but not explanatory) depending on the SRS in which it is proved and the SRS in which the prover is considering the proof. We explore these ideas further in the following sections.

We envision that different proof-related activities occur within a given SRS. Proofs may be direct or indirect, and it may be the case that multiple different proofs could exist within each SRS. Further, proofs within a given SRS may be formal or informal, and the given SRS determines what rules, tools, approaches, and conventions apply to the given SRS. As noted, we also view that there is potential movement between SRSs.

Insights from proofs of binomial identities

---

[1] Also, while Weber (2010) defined an explanatory proof from the perspective of proof comprehension (talking about a reader of a proof), we could similarly consider an explanatory proof from the perspective of proof production. That is, the proof that has been produced may be explanatory if it enables the *prover* of the proof to translate the argument that he or she is formulating to an argument in a separate semantic representation system.

In order to elaborate these ideas, we provide examples from combinatorics, specifically proving binomial identities. There is nothing in the notion of SRS that specific to combinatorics (which we address in the Discussion Section). But, we contend that proofs of binomial identities are particularly enlightening because combinatorics naturally lends itself to moving between semantic domains. Indeed, as we will describe below, it is commonplace to use another SRS (perhaps an algebraic system) to prove a relationship in a given SRS (perhaps an enumerative system). In the following section we will provide an algebraic and an enumerative proof of the statement $\binom{n}{k} = \binom{n}{n-k}$, acknowledging that we could also explore additional SRSs of this same expression (such as induction or block-walking). We will present these each of these proofs through the lens of a different SRS.

An explanatory proof in the algebraic SRS

In the algebraic SRS, $\binom{n}{k} = \binom{n}{n-k}$ can be interpreted as a statement about (nonnegative) integers, and valid tools include properties of integers and algebraic rules. Substituting the definition of binomial coefficients ($\binom{n}{k} = \frac{n!}{(n-k)!k!}$) into the identity and applying rules of algebra yields the following proof:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{\big(n-(n-k)\big)!\,(n-k)!} = \binom{n}{n-k}$$

Since we can use rules of algebra to manipulate one expression into the other, both sides of the equation are equivalent, and so the statement is true. We call this an explanatory proof in the algebraic SRS (or an algebraic proof of the identity) because it follows algebraic rules to demonstrate why the identity is true.

An explanatory proof in the enumerative SRS

There are also enumerative, or combinatorial, proofs to this identity. In an enumerative proof, we argue that the two sides of the identity each represent two different counting processes (in the sense of Lockwood, 2013) that either a) count the same set of outcomes (a direct combinatorial proof) or b) count two different sets of outcomes between which there is a bijection (a bijective combinatorial proof). For the sake of simplicity, we give one example of a direct combinatorial proof, noting that there are many other enumerative proofs we could introduce. Note that these enumerative proofs look quite different than the algebraic proof presented above, and sentence descriptions of counting processes and sets (rather than manipulation of algebraic symbols) comprise the proof.

We show that both sides of the identity count the number of *k*-element subsets of an *n*-element set. That is, we interpret $\binom{n}{k} = \binom{n}{n-k}$ as being a statement that relates different kind of subsets of *n*-element sets. The left-hand side counts this set by selecting *k* elements from *n* distinct elements that should be included in the subset, and this process reflects the left-hand expression of $\binom{n}{k}$. The right-hand side counts this set by using the notion of a complement of the set – by selecting the *n-k* elements from *n* distinct elements that should *not* be included in the subset. Therefore, because both sides of the identity count the same set, they represent expressions that are numerically equal, and thus the equality holds.

Explaining and convincing in algebraic and enumerative proofs – what is being explained, and what is convincing?

We take these two proofs to further our discussion about proofs that explain versus only convince. The perhaps "easy" way to interpret these two proofs in terms of proofs that explain and proofs that only convince is to say that the algebraic proof convinces but does not explain, while the enumerative proof is somehow more explanatory. However, we argue that there is a deeper story to tell, and each of the above proofs could be considered to be explanatory and/or convincing depending on which SRS we are considering.

In particular, we contend that the question *What is the proof explaining?* is not a simple inquiry. We argue that the enumerative proof is explanatory in the SRS of enumeration because it demonstrates *why* both sides of the identity counts the same set of outcomes. Further, following our definition of proofs that explain (which we borrow from Weber, 2010), there is a particular instantiation to properties that we know about sets and choosing elements of sets that makes a meaningful connection between the expressions, the counting processes described in the proof (Lockwood, 2013), and what we know about what it means for sets to be equal. Thus, this proof satisfies our need for understanding what is happening enumeratively. However, the enumerative proof is *not* explanatory in the SRS of algebra. That is, the enumerative proof does nothing to explain why the identity holds algebraically.

Conversely, it is true that the algebraic proof does not provide any explanation for why the identity is true in the enumerative domain. However, we claim that the algebraic proof *is* explanatory in the SRS of algebra. Specifically, using the definition of binomial coefficients and the rules of algebra we can see the logical, algebraic steps that justify why that relationship is true. Thus, we could say that that proof explained why, algebraically, the relationship holds.

We claim that if a proof is explanatory in a given SRS it is necessarily convincing, but a proof may be convincing but not explanatory for a different SRS. Returning to our examples of algebraic and enumerative proofs of $\binom{n}{k} = \binom{n}{n-k}$, we would say that the algebraic proof may be convincing in the enumerative system, even if it not explanatory in that enumerative system. Similarly, the enumerative proof may convince someone that the algebra must be true, even if the enumerative proof offers no insight into why the algebraic steps are true.

We commonly use this relationship between SRSs in proving theorems and identities in combinatorics. To emphasize this point, consider the relationship $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, which we mentioned in the introduction. This is an identity that is quite natural to prove enumeratively.[2] However, it is not immediately apparent why the algebra should hold. Summing all of the terms, finding common denominators, canceling, and simplifying for the general value of *n* require considerable work, particularly by hand. Here, then, the enumerative proof may *convince* us of the algebra, even if we cannot actually describe and list out all of the steps that would satisfy the identity algebraically. If all we needed was to be convinced that this identity holds, it would make sense to use a combinatorial argument to prove the result, rather than an algebraic one.

More commonly in combinatorics research, we go in the other direction – we use algebra to convince us of identities that are difficult to prove combinatorially. For example, generating functions (e.g., Wilf, 2005) offer a well-established technique of translating difficult combinatorial questions into more manageable algebraic settings. In this technique, we encode combinatorial objects as coefficients of polynomials, and we use rules of polynomials and algebra to derive results that are then translated back to the combinatorial context. A proof of an

[2] Both sides count the total number of subsets of any size from a set of *n* elements. The left-hand side counts this by summing up all possible numbers of k-element subsets for values of k from 0 to n. The right-hand side counts this by considering, for each of the n elements in the set, whether or not it is an element of a subset.

identity involving generating functions is explanatory in an algebraic domain, as it demonstrates clearly why the algebra holds to establish the relationship, but it does not explain why the relationship holds from an enumerative perspective. The fact that we have different SRSs in which to have proofs convince or explain is a wonderful aspect of mathematics, as it opens up many opportunities for us to develop convincing proofs even if one SRS is particularly difficult.

Our point, then, is that it sells proof short to simply characterize a proof as being convincing or explanatory without further specifying what precisely is being explained. Further, it is misleading to dismiss algebraic or inductive proofs as being necessarily not explanatory. Certainly, progress is being made in this regard (e.g., Stylianides, et al. (2016)), and we want to contribute to these conversations about what it can mean for a proof to be explanatory.

Further, returning to an important distinction in the proof literature, SRSs also allow us to address another dimension of this conversation – the importance of who the prover (or the reader) is. As noted above, we particularly appreciate Weber's (2010) viewpoint in clarifying that these ideas must be considered from the prover's/reader's perspective, and we also adopt this framing. That is, different proofs may be explanatory or convincing in different SRSs depending on the perspective of the prover. Weber's (2010) notion of SRSs is in line with this perspective, and we note that for an individual prover (or reader), he or she may naturally tend toward a particular SRS. Based on a person's background or familiarity with ideas (their concept image), they may be more or less inclined to be able to deem a certain proof as explanatory or convincing, depending on which system they are examining.

Discussion and Conclusion

In this paper, we have argued that combinatorics (and proofs of binomial identities) offers a novel mechanism by which to investigate proofs that explain versus proofs that only convince. In this section, we highlight points of discussion and implications related to this conversation.

Combinatorics as a rich domain in which to study proof

Combinatorics is a fertile domain in which to study proof. In particular, binomial identities (and combinatorics more generally) are characterized by translation between SRSs, and this has repercussions for elaborating the ideas of proofs that only convince and proofs that explain. We hope that more proof researchers will explore this domain, as it may potentially shed light on other interesting aspects of proof.

The discussion in this paper extends to other mathematical domains

And yet, even though we want to make a case for the value of combinatorics in studying proof, our findings and discussion are not unique to combinatorics. Although we have primarily focused on combinatorial examples of proving binomial identities to discuss SRSs and proofs that convince and explain, these ideas also extend to non-combinatorial contexts. For instance, we could consider different proofs of the Pythagorean theorem. A proof without words (Nelsen, 1993) of the Pythagorean theorem may be explanatory in a geometric SRS, but it may not be explanatory in the algebraic SRS. Similarly, in an algebraic proof, if the numbers are viewed only as integers and not as side lengths with some dimension, then the algebraic manipulation is explanatory in the algebraic system but not in the geometric system.

Pedagogical suggestions

We have demonstrated the value of translating between SRSs, and we have shown that in some fields like combinatorics this is a natural thing to do. However, we want to emphasize that we should be careful when translating between SRSs. For example, when translating between an algebraic and an enumerative SRS when proving a binomial identity, one must consider what assumptions can be made within a given SRS.

The notion of SRSs in proof also allows us to reframe how we think about students' proving activity. The idea that students might be working from different SRSs gives a useful lens through which to consider student activity in discrete math classes, perhaps giving students more credit than simply interpreting their activity as meaningless and purely syntactic. When a student tends toward algebra when proving a binomial identity, it is easy to assume that the student is being shallow (and we admit to adopting this perspective at times). However, such a student may be viewing the statement to be proven through an algebraic SRS, which may be meaningful to them. Thus, this perspective on proofs that convince versus explain may give agency to the prover.

Finally, for those teaching discrete math, we suggest to keep in mind that in teaching counting and binomial identities specifically, we are asking students to coordinate multiple SRSs. The notion of SRSs highlights the fact that any discussion of explaining and convincing must be considered from the perspective of the prover and/or reader. In combinatorics, there are many different perspectives from which to interpret/view the same symbolic binomial identity. The fact that so many different SRSs exist (particularly in proving binomial identities) highlights that it is important to consider who is proving or reading a proof.

References

Bell, A. W. (1976). A study of pupil's proof explanations in mathematical situations. *Educational Studies in Mathematics, 7*(1), 23-40.

De Villiers, M. (1999). The role and function of proof. In M. de Villiers (Ed.) *Rethinking Proof with the Geometer's Sketchpad* (pp. 3-10). Key Curriculum Press.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*, 5-23. DOI: 10.1023/A:1012737223465.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*(4), 389–399.

Nelsen, R. (1993). *Proofs Without Words: Exercises in Visual Thinking*. Washingtong, DC: MAA.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*(3), 319-325.

Rav, Y. (1999) Why do we prove theorems? *Philosophia Mathematica, 7*(3), 5–41.

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies, 34,* 135-151.

Stylianides, G. J. (2009). Reasoning and proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11,* 258-288.

Stylianides, G. J., Sandefur, J., & Watson, A. (2016). Conditions for proving by mathematical induction to be explanatory. *The Journal of Mathematical Behavior, 43*, 20-34.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Weber, K. (2010). Proofs that develop insight: Proofs that reconceive mathematical domains and proofs that introduce new methods. *For the Learning of Mathematics, 30*(1), 32-37.

Weber, K. & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56,* 209-234.

Wilf, H. (2006). *Generatingfunctionology* (3rd ed.). Wellesley, MA: A.K. Peters, Ltd.