What is Difficult About Proof by Contradiction?

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Although students face many challenges in learning to construct mathematical proofs in general, proof by contradiction is believed to be particularly difficult for them. We investigate whether this is true, and what factors might explain it, using data from an "Introduction to Proof" course. We examined proofs constructed by students in homework and examinations, and conducted stimulated-recall interviews with some students about their thought processes while solving proof problems. Preliminary analysis of our data suggests that students' background knowledge about the typical content domains that appear in indirect proof plays a larger role than the logical structure of the proof technique itself.

Keywords: Indirect Proof, Proof by Contradiction, Teaching Mathematical Proof.

Introduction

The teaching and learning of mathematical proof have received a great deal of attention in the field of mathematics education research, and this emphasis continues to increase. At the college level, the ability to understand and construct proofs is essential for students to transition from computationally oriented calculus sequences to more theoretically oriented upper-division mathematics courses. Many universities have instituted "Transition to Proof" courses to facilitate this. At the K-12 level, the Common Core Standards for Mathematical Practice emphasize the ability to construct and critique mathematical arguments, i.e., proofs.

Indirect proof, also known as proof by contradiction (we will use these terms interchangeably), is an essential form of proof across all mathematical content areas. Instructors' anecdotal experience as well as mathematics education research suggest that students have particular difficulty with this type of proof, where "difficulty" has been variously interpreted as pertaining to comprehending, constructing, deriving conviction from, or simply disliking indirect proofs (Tall 1979, Brown 2018). It is somewhat puzzling why indirect proof would be especially challenging cognitively, given that we use this kind of informal reasoning frequently in everyday life (Reid & Dobbin 1998). The common form of argument, "If that were true, then how do you explain X?" is clearly an informal sketch of a proof by contradiction.

If indirect proof is indeed uniquely difficult in the formal mathematical context, what are the reasons for this? We explore this question using data from an "Introduction to Proof" course recently taught by one of us, in the context of students *constructing* indirect proofs for homework assignments and examinations.

Purpose and Theoretical Background

The purpose of this study was to identify difficulties faced by students in constructing indirect proofs as part of their regular coursework in an "Introduction to Proof" class. The problems solved by these students are part of the regular course pedagogy rather than tasks chosen by a researcher, for example in an interview setting. Since this study is somewhat exploratory, we wanted a wide range of "naturalistic" proof samples rather than one or two that might reflect peculiarities of those chosen tasks more than general student issues with indirect proof.

Assuming that the literature is correct that indirect proof is uniquely difficult for students, our intent was to test three hypotheses that might explain why, or to formulate additional ones.

1. *Logical hypothesis*. Students have difficulty recognizing what constitutes a contradiction in the strict logical or mathematical sense. If a step in their proof contradicts a piece of their prior mathematical knowledge (which may never have been rigorously proved itself), is that sufficient? They may also manipulate mathematical statements too formally, assigning them so little *meaning* that contradictions go unrecognized (Sierpinska 2007).
2. *Psychological hypothesis*. Indirect proof requires the temporary acceptance for the sake of argument of assumptions that are actually false, and may already be known to be false. Such *counterfactual reasoning* may be more difficult within the domain of mathematics than in everyday contexts (Antonini & Mariotti 2008). For example, I can fairly easily imagine that Hillary Clinton won the 2016 election, but how can I imagine that 7 is not a prime number? What kinds of reasoning can be trusted in such an "impossible world"?
3. *Structural hypothesis*. In a direct proof task, both the hypothesis and the conclusion are known at the outset. That is, one knows where the proof begins and where it will end, providing a structural framework (Selden & Selden 2009). In contrast, the goal of indirect proof is "a contradiction". The prover does not know in advance what this will be, so cannot structure the proof around it.

Our initial research question was, what evidence do students' proofs from their class assignments provide for or against these hypotheses?

Based on our initial data analysis, however, we have broadened our hypotheses. It may be that indirect proof is difficult not (only) because of its logical *nature*, but because of the typical mathematical *content* in such proofs, for example rational versus irrational numbers. The background knowledge and beliefs that students have about such content may be a source of their difficulties. One would then expect to observe similar difficulties in direct proofs dealing with the same content. We have found it useful to think about students' background or prior content knowledge in terms of the *resource framework* (Hammer et al 2005), or the *knowledge-in-pieces* viewpoint (diSessa 2013). In these perspectives, student knowledge does not form a coherent theory, but rather a collection of pieces or "resources" that may not be mutually consistent and may be individually activated in varying circumstances. From such a perspective one would not ask whether a student "really believes" for example that $(a+b)^2=a^2+b^2$ but rather in what contexts this type of assertion is activated.

Participants and Methods

The participants in this study were students in an "Introduction to Proof" class taught by one of the authors at a large public university in the southwestern United States. The class is normally taken following the two-year calculus sequence and is required for all mathematics majors. Of the 106 enrolled students, 72 agreed to participate in the study. The majority of these were mathematics majors, and the rest were from various other STEM majors. There were roughly equal numbers of male and female students.

All homework and exams were graded using the Gradescope system, which preserved the students' work for our later analysis. There were twelve graded proof by contradiction problems, some from the course textbook and some that we added based on previous research or for pedagogical reasons. During the course no attempt was made to match the assigned direct and indirect proof problems for difficulty or content, but for our analysis we selected a comparison group of eleven direct proof problems that we considered comparable in difficulty and subject matter. The comparison is quite rough, since the direct proof problems assigned tended to involve specific topic areas covered in the course, such as equivalence relations or mathematical induction, which do not overlap greatly with the content areas for the indirect proofs. The proof by contradiction problems were clustered in two consecutive homework assignments near the middle of the course, or on the second midterm or final exam. We also solicited student volunteers to be interviewed, but only obtained six volunteers, all of whom were accepted. Nevertheless there were three male and three female interview subjects, representing a range of achievement levels in the course.

Interviews took place just after the second midterm exam. These were semi-structured "stimulated recall" interviews (Shubert & Meredith 2015). Students were shown their own prior work on certain indirect proof problems, and were asked to identify the contradiction they reached and explain why it was a contradiction, how they searched for and then recognized the contradiction, why they chose a particular approach, and what other approaches they had attempted. Sometimes they were shown the work of another student and asked to locate the contradiction or to compare that solution with their own. After discussing specific proof problems, they were asked some general questions, such as what makes an indirect proof work, how they feel about reasoning on the basis of a counterfactual assumption or "impossible" geometric diagram, and whether they prefer direct or indirect proof for any reason.

Homework and examinations are complementary data sources in some respects. Students are under less time pressure when solving homework problems, so one might expect their reasoning to better reflect their capabilities and knowledge about proof rather than careless errors due to time pressure. On the other hand, students have more opportunities to obtain help from friends, teachers, or online sources, when doing homework. We saw evidence for both effects.

At this stage of data analysis, we have examined all student solutions to six indirect and two direct proof problems. We coded the different approaches taken, both correct and incorrect, and created categories of errors or misconceptions exhibited. Of particular interest were the types of contradictions obtained, whether they were reached in an efficient or a roundabout manner, whether any actual contradictions were written down but overlooked by the student, or conversely whether a student claimed to have reached a contradiction when she had not in fact deduced one. We have transcribed the interviews and begun to code them for students' understanding of how proof by contradiction works, ability to recognize contradictions, comfort level when reasoning from counterfactual hypotheses, and so forth.

Results

As an initial rough indication of whether the indirect proofs were "more difficult" than the comparison group of direct proof problems for our students, we compared their mean scores on the two groups of problems using the Welch two-sample t test. The difference in group means was not significant at the 5% level, suggesting roughly similar levels of difficulty.

One of the homework problems assigned was the Angle Bisector problem studied by Baccaglini-Frank et al (2013): show that the bisectors of two angles in a triangle ABC cannot be perpendicular to one another. This is an easy consequence of the angle sum in a triangle being 180 degrees and can be demonstrated by direct (6 students) or indirect proof (48 students). Significantly, most students included a diagram with their proof and showed no confusion in reasoning from this impossible geometric figure (termed a *pseudo-object* by Baccaglini-Frank et al). Asked how he felt about this potential cognitive conflict, one student explained:

> I think it might just be from experience of knowing that hand-drawn pictures can be inaccurate, and then there are a lot of stuff like optical illusions where some things look perpendicular when they're not…It wasn't necessarily to reassure myself that the statement was true because I knew the statement was true, and it was more so to visualize the relationship of that new angle and how it relates to A,B, and C.

This and similar data do not support our Psychological hypothesis.

Another homework problem asked students to prove that no positive integers m and n satisfy the equation $7/17 = 1/m + 1/n$. There are many ways to show this, but only 8 out of 64 submissions were correct. The most common approach was to write $7/17 = (m+n)/mn$, or the equivalent form $7mn = 17(m+n)$. 24 students concluded incorrectly from this that $m+n=7$ and $mn=17$. We coded this reasoning, which we also observed in other problems, as SFE (Strong Fraction Equivalence, the view that equal fractions must have identical numerators and denominators) or SUF (Strong Unique Factorization, the view that $ab=cd$ implies that $a=c$ or $a=d$). These students then easily showed that the few values for m and n consistent with one of these restrictions do not satisfy the other. SFE might reflect students' uncertainty about when it is legitimate to assume that a fraction is in lowest terms (as is often done "without loss of generality" in indirect proof) but this explanation does not seem to account for SUF.

It is not plausible that large numbers of STEM majors "really believe" SFE or SUF, certainly not as part of a coherent set of background beliefs, but clearly these assertions are commonly activated (in direct proofs as well). This and similar observations of ours make more sense in terms of the resource framework. Neither SFE nor SUF seems connected to the logical structure of indirect proof, but the typical content of this type of proof may provide more opportunities than that of direct proof to activate such resources. These observations are consistent with the Logical hypothesis, in the sense that students are working formally rather than attending to the meaning of their mathematical assertions. However, we think it is more productive to locate their difficulties in the content of the proof (integers, rational numbers, divisibility) than in the logical structure of indirect proof.

SFE and SUF can be viewed as parts of a broader category of errors that we observed in our data and termed AVNP, for Algebraic Visibility versus Numerical Possibility. That is, students attend to what is algebraically visible in an equation rather than what numerical possibilities might be consistent with it. For example, students see that an expression is not an algebraic perfect square, so they assert that it cannot be a square for any specific integer values of the variables. Or, a rational expression is in lowest terms (the numerator and denominator have no algebraic common factor) and students assume that it must be in lowest terms for any specific integer values of the variables. Examples occurred in both direct and indirect proofs. The domains of variables are also not explicitly visible in the expressions that students manipulate, and we observed uncertainty about the properties and relationships of the domains $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$. For example, concepts like divisibility that only apply in $\mathbf{Z}$ were used in $\mathbf{Q}$ or $\mathbf{R}$, as has been previously observed (Barnard & Tall 1997).

Conclusion

Our initial analysis of the data suggests that our students have a rather good understanding of and comfort level with proof by contradiction. They can negate claims, identify contradictions, and explain the logic of indirect proof, and they seem generally unperturbed by counterfactual reasoning. The difficulties they encounter in solving proof problems seem to reflect the subject matter of the proof more than the proof type (direct or indirect). Many of the "misconceptions" they exhibit cannot be understood as genuinely held beliefs, which supports viewing them as resources or pieces of knowledge that are activated in particular contexts. The "difficulty" of proof by contradiction may lie in the types of resources that it tends to activate.

Discussion Questions

1. How can we operationally distinguish proof errors that reflect difficulty with indirect proof as such from those that reflect misconceptions about the subject matter of the proof, for example the rational number system?
2. How can we better understand student "misconceptions" as resources activated in specific contexts?
3. How can we improve the design of this study for future replications or extensions?

Acknowledgment

We thank Amelia Stone for assistance in conducting and transcribing the interviews.

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