# Abstract Algebra Instructors' Noticing of Students' Mathematical Thinking

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*Examining teaching practices in advanced mathematics is a relatively new field of scholarship despite a long history in K-12 settings. One important research area in this setting is documenting teacher noticing of students' mathematical thinking. In this report, we extend this line of work to explore how undergraduate mathematics instructors attend to, interpret, and respond to student thinking (Jacob, Lamb, & Philipp, 2010) in abstract algebra. We surveyed 25 abstract algebra instructors with a range of experience. Overall, we found that our participants focused on student thinking to a greater degree than the elementary teachers in earlier studies. Further, their interpretations spanned two distinct foci: understanding of concepts and the formal representation system. Their proposed responses then reflected a wide span of teaching actions. This exploratory analysis unveiled a number of previously undocumented characteristics of instructor noticing at the undergraduate level which can serve to inform future research on teaching practices.*

*Keywords:* Abstract Algebra, Teaching, Noticing

The mathematics education community has moved towards models of teaching where instruction is tied to students' mathematical thinking (Jacobs & Spangler, 2018). Jacobs and Spangler identified teacher noticing as one of two core instructional practices needed for this type of instruction. While this construct has been studied and unpacked from a multitude of lenses at the K-12 setting (e.g., Jacobs, Lamb, & Philipp, 2010; Sherin & van Es, 2005; Star & Strickland, 2008), very little is known of instructor noticing at the undergraduate level. A lack of research in this setting is unsurprising in light of Rasmussen and Wawro's (2018) recent look at research at the post-calculus level where teaching is just beginning to be studied.

In this report, we share results from an exploratory study unpacking a particular aspect of instructor practice: noticing of students' mathematical thinking. We adapt this lens from the work of Jacobs, et al. (2010) who have decomposed teachers' in-the-moment noticing into three related acts: (a) *attending to children's strategies*, (b) *interpreting children's understandings*, and (c) *deciding how to respond on the basis of children's understandings.* Our study is situated in the context of abstract algebra, a standard upper level undergraduate course. We leverage pieces of student work that reflect documented ways students reason about the core concepts of identity, subgroups, and cyclic groups. Through surveying a variety of instructors, we introduce analysis of how mathematics instructors are attending to, interpreting, and responding to the student responses. We pay particular attention to how these responses diverge from the responses documented in the K-12 literature in order to contribute to our knowledge of teaching practices at the advanced undergraduate level.

## Background

In this section, we provide background both on noticing research at the K-12 level and the larger research base on teacher practices at the advanced undergraduate level.

### Noticing at the K-12 Level

Noticing student thinking is a "core practice of high-quality mathematics instruction because it is foundational for teachers' in-the-moment decision making" (Jacobs & Spangler 2017, p. 192). Which aspects of student thinking teachers give their attention to and how they interpret what they see or hear, influences their instructional decisions (Jacobs et al., 2010; Schoenfeld, 2011). Researchers have documented that teachers and prospective teachers notice a multiple of things when engaging with videos of classrooms (e.g., Sherin & van Es, 2006; Star & Strickland, 2008). Jacobs and colleagues (2010) developed a framework to distill one aspect of this noticing: noticing students' *mathematical thinking*. This framework can be leveraged to explore teacher noticing in the context of written artifacts or short video clips of students engaged in mathematical tasks. As teachers engage in describing, interpreting, and deciding how to respond to artifacts of student work, they demonstrate their skill in noticing mathematical thinking.

A number of researchers have built off of this work from the elementary level to study varying populations of teachers including Simpson and Haltiwanger's (2017) recent work at the secondary level. As noted by Nickerson, Lamb, and LaRochelle (2017), expanding beyond the elementary level brings additional challenges including the availability of artifacts, the availability of well-articulated frameworks around student thinking, and the availability of expert responses. Such work may also require adaptations to the original framework in light of the new contexts (see Simpson & Haltiwanger's additional distinctions.)

The summative results from these studies reflect that (a) professional noticing of student thinking is an essential skill for teachers and (b) it is a skill that can be developed through appropriate support (Fernandez, Llinares, & Valls, 2013; Jacobs, Lamb, & Philipp, 2010; Miller, 2011). Examining noticing at the advanced undergraduate level likely requires both consideration to the elementary literature base and consideration of what aspects of noticing may be informed by the particulars of the advanced mathematics context.

**Teaching at the Advanced Undergraduate Level**

Few studies at the advanced undergraduate level have focused "directly on *teaching practice*—what teachers do and think daily, in class and out, as they perform their teaching work" (Speer, Smith, & Horvath, 2010, p. 99). A few exceptions have begun to unpack some of the relevant practices including the nature of lectures (Weber, 2004), question types in lecture (Paoletti, Krupnik, Papadopoulos, Olsen, Fukawa-Connelly & Weber, 2018), and grading student proofs (Moore, 2016). Little work has explored the nature of teaching practices directly related to engaging with students and their thinking. The studies that have begun unpacking this work are situated largely in the implementation of inquiry oriented-instruction (IO), a pedagogy that relies heavily on instructor use of student ideas as a component of lessons aimed to move from informal to formal understanding of ideas (Rasmussen & Wawro, 2018).

Instructors using both the differential equations IO materials and abstract algebra IO materials have been documented to struggle to support productive discussions and leverage student reasoning without strong pedagogical content knowledge (Speer & Wagner, 2009; Johnson & Larsen, 2012). While mathematician instructors likely have very strong mathematics content knowledge, their knowledge of student reasoning and connections to pedagogy may not be as fully formed. Pedagogical content knowledge provides the lens through which instructors can interpret and respond to student thinking. In this way, teacher noticing is a specific practice or skill, related to pedagogical content knowledge, that becomes critical for instructors striving to adjust their lessons based on student thinking, as is often the case in IO classrooms.

Johnson and Larsen (2012) and Johnson (2013) provide perhaps the most nuanced look of addressing and leveraging (or failing to leverage) student reasoning in advanced undergraduate settings through their look at IO curriculum implementation in the abstract algebra classroom. In particular, Johnson and Larsen highlight the role of *generative listening*. This listening occurs when an instructor is able to interpret students' reasoning and adjust the lesson trajectory accordingly. Johnson and Larsen noted that their case study instructor often lacked knowledge of the specifics of student reasoning such as seeing operating on symmetry elements left-to-right, and thus failed to appropriately respond. In Johnson's follow-up work, she provides contrasting images of abstract algebra instructors' productive mathematical activity that was needed to interpret and analyze student ideas, as well as make connections between these ideas and the larger mathematical goals of the lessons. These studies provide cases that establish the important role of noticing student reasoning in order to promote student-centered instruction. They also illustrate that the knowledge and skills involved in supporting students in abstract algebra is non-trivial.

## Theoretical and Analytic Orientation

Our work is orientated towards teaching practices, the work teachers do in their daily lives as instructors (Speer, et al., 2010). In particular, we focus on their noticing of student thinking, and ultimately the nature of the responses connected to this noticing. We make the assumption that "teacher noticing is worthy of study because teachers can be responsive only to what has been noticed" (Jacobs & Spangler 2017, p. 192). We leverage the framework introduced by Jacobs, Lamb, and Philipp (2010) that unpacks noticing as three interrelated practices: (a) *attending to children's strategies*, (b) *interpreting children's understandings*, and (c) *deciding how to respond on the basis of children's understandings.* Each practice can range from noticing that is disconnected from students' thinking to noticing meaningfully and richly coupled with student thinking.

Beyond the scope of the original framework, we incorporate other theoretical distinctions to produce a more detailed image of instructors' interpretations and responses to students. First, at this level, interpreting can have both a semantic orientation, focused on concept understanding, and logio-structural (formal) orientation, focused on aspects of proof and the formal structure emphasized in advanced mathematics (c.f., Weber, 2004). We also parsed the nature of responding to not just how coupled the response was with student thinking, but also the nature of the response itself--what did these instructors say they would do next with this student? Responding is a practice that has substantial theoretical breakdowns at the K-12 level (e.g., Boaler & Humphreys, 2005; Herbel-Eisenmann, Drake, & Cirillo, 2009; Milewski & Strickland, 2016) focused on the nature of teacher questions and actions. To analyze our instructor responding moves, we leveraged various literature to identify key ways of responding in terms of question types (Sahin & Kulm, 2008), and other responding moves (Milewski & Strickland, 2016). We expand upon our categories in the next sections.

## Methods

For this study, we surveyed 25 Abstract Algebra instructors representing a range of experience and institution types. Table # reflects the demographic information of the participants.

*Table 1. Background on Participants.*

| Research Focus | | Experience Teaching Algebra | | Position | | Institution Type (highest mathematics degree) | |
|---|---|---|---|---|---|---|---|
| Abstract Algebra | *n*=11 | <5 times | *n*=8 | Assist. Prof. | *n*=2 | Ph.D. | *n*=10 |
| Math Education | *n*=5 | 5 - 9 times | *n*=10 | Assoc. Prof. | *n*=9 | M.S. | *n*=8 |
| Other Math Pure | *n*=9 | > 9 times | *n*=7 | Full Prof. | *n*=11 | B.S. | *n*=6 |
| | | | | Other | *n*=3 | NA | *n*=1 |

**The Survey**

The survey was directly adapted from Jacobs, Lamb, and Philipp (2010) and Jacobs, Lamb, Philipp, and Schappelle (2011). The instructors were given five pieces of student work. For each piece of student work, instructors were asked:

- Please describe in detail what you think this student did in response to this prompt.
- Please explain what you learned about this student's understanding.
- Pretend that you are the instructor of this student. Describe some ways you might respond to this student, and explain why you chose those responses.

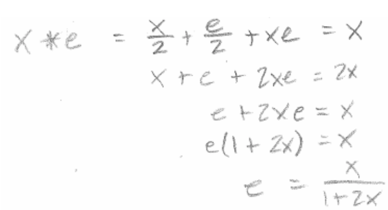
The student work stemmed from a large-scale project collecting data about student understanding in group theory (Melhuish, 2015). Table 2 contains three pieces of student work that are the focus of this report.

*Table 2. Sample Student Work Provided to Participants*

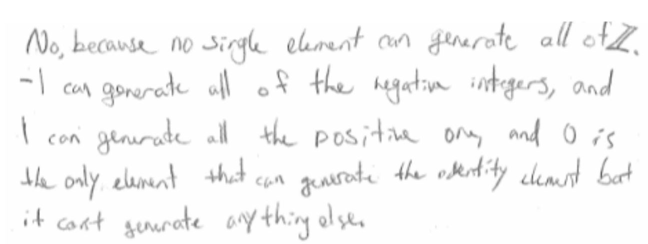
(1) Given L the set of all positive rational numbers, consider the binary * defined:

x*y=x/2+y/2+xy

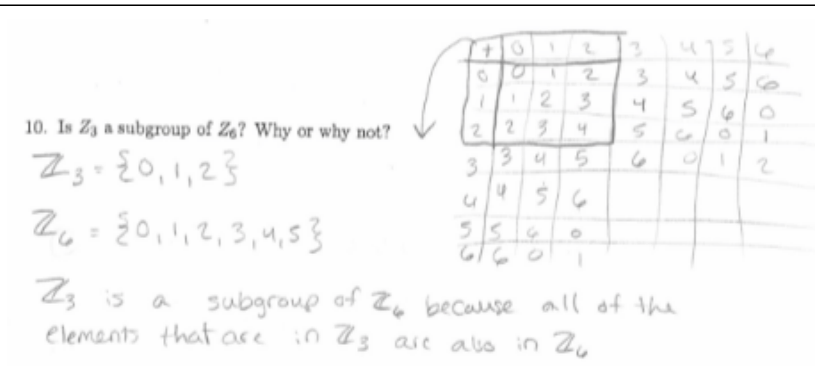
Determine if this operation has an identity. If so, identify the identity.




(2) Is *Z*, the group of integers under addition, a cyclic group?




(3)



Each response was selected due to its connections to established ways of thinking about group theory topics from the literature. Response one stems from a task identified by Novotná and Hoch (2008) as reflecting structure sense for operation where students may or may not recognize that an identity element must serve as an identify for all elements in the set. The second piece of student work reflects incomplete coordination of the binary operation with subgroups where students may rely on a subgroup test without attending to the differing operations between $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_6$ (e.g., Melhuish, 2018; Dubinsky, Dautermann, Leron, & Zazkis, 1994) The third piece of student work reflects a common conception of cyclic groups where elements only generate via repeated operation and thus do not take on negative powers (Melhuish, 2018; Lajoie & Mura, 2000).

**Analysis**

To analyze the data, we incorporated a two-fold approach. First, we analyzed the data using the original scheme developed by Jacobs, Lamb, & Philipp (2010) addressing whether instructors attended to student thinking, the robustness of their evidence of interpretation, and the degree to which their responding actions were connected to the student's thinking. Second, we took a more grounded approach to account for the fact that the advanced tertiary level may lead to substantially different characteristics of instructor noticing. The initial passes were done by the authors independently. In collaboration, we then arrived at a coding scheme where all responses were classified. A subset of the scheme can be found in Table 3. All instructor responses were then coded in tandem with discussion serving to settle disagreement in codes.

*Table 3. Background on Participants.*

| Noticing Practice | **Categories** (*Codes*) |
|---|---|
| Attending | **Connected to Student Thinking** (*Y:Yes*, *N:No*) |
| Interpreting | **Evidence Level** (*N: No Evidence Provided*, *L: Limited evidence provided*, *R: Robust Evidence Provided*); **Aligned with Literature Interpretations** (*Y:Yes*, *N:No*); **Formal Representation System** (*D: Definition*, *A: Implicit Assumptions*, *P: Proof*, *Q: Quantifiers*) |
| Responding | **Connected to Interpretation of Student Thinking** (*Y:Yes*, *N:No*); **Nature of Response** (*E:Praise*, *T:Telling*, *G:Guiding*, *P:Probing*, *C:Command*) |

## Results

**Attending & Interpreting**

The abstract algebra instructors provided quite different profiles in terms of noticing. Compared to the documented literature on elementary teachers, these instructors were much more likely to attend to students' thinking. A typical interpretation looks as follows:

> The student understands which binary operation is in question. The student has done some elementary algebra correctly, from which the question could be answered. But I would infer from the response stopping at this point, that the

student thinks that there is an identity and that it has been found (referencing task 3).

In fact, across our three focal tasks, we documented zero instances of not paying attention to students' thinking and only 7% of responses provided largely evaluative statements. For example, one instructor made comments such as, "[I]mpressive written response, a good 'abstract algebra' presentation..." Such a response illustrates a focus that was more evaluative with language about the quality of the response, and less focus on the student thinking offered. While the instructors did largely attend to student thinking, there was range of evidence provided as (see Table 4).

*Table 4. Percentage of Interpretations with Particular Characteristics*

| Task | Level of Evidence Provided | Aligned with Literature | Focused on Aspects of Formality |
|---|---|---|---|
| Task 1 (Identity) | R: 12% L:40% N:48% | Y:48% | Y: 60% |
| Task 2 (Subgroup) | R: 38% L:48% N:14% | Y:65% | Y: 54% |
| Task 3 (Cyclic Group) | R: 14% L:52% N:38% | Y:73% | Y: 60% |

One feature that differed across our participants was attention to aspects of the formal representation system which split the interpretations. Non-formal interpretations included statements like "confused the notions of subset and subgroup." For formal interpretations, the role of definition was particularly prominent (52%[1]) (e.g., "the student does not appeal to the literal definition"), followed by hidden assumptions (14%) (e.g., "The student assumed the distributive law holds."), quantifiers (14%) "e.g., "Student has a weak grasp of the words 'for all'.", and issues of proof (12%) (e.g., "[I]t does [not] formally 'prove' that an identity element exists").

**Deciding How to Respond**

Deciding how to respond to a student is an important aspect of noticing, but is also its own area of research in K-12 scholarship on teaching with little corresponding research at the tertiary level. In terms of the original noticing framework, we note that almost all of the mathematician responding choices were connected to their interpretations of student thinking (97%). This attention was a dramatic shift from what was documented with elementary teachers where a sizeable portion did not attend to student thinking (Jacobs, et al., 2010).

*Table 5. Percentage of Response Types*

| Type | Example Response | % [1] (*n*=72) |
|---|---|---|
| Command | "Please review the definition of subgroup which has three parts (closure, identity, inverses). Then come to my office hours." | 8% |
| Praise | "First is praise the amount of good work that happened." | 8% |
| Probe | "Tell me your reasoning for what you did? What does this answer | 17% |

[1] Interpretations could include multiple components. Percentages are not intended to sum to 100%

| | | |
|---|---|---|
| | mean?" | |
| Tell | "I remind you that identity cannot depend on $x$" | 35% |
| Guide | "I think this student is ready for direct questions about their solution. For example: Is it a problem that $x$=1 and $x$=2 result in different values for $e$?" | 65% |

1. Percentages sum to greater than 100% because some responses included multiple response types

Even though responses were connected to student thinking, the nature of the responses varied. Guiding questions, questions intended at move students' mathematics to the correct mathematics were by far the most prevalent for these instructors. However, there was also substantial telling responses along with commanding, praising, and probing. See Table 5 for examples of each type of response and the respective percentages.

**Discussion**

The abstract algebra instructors provided a contrasting profile to what has been documented about teachers at the K-12 level. First, with a few exceptions, the abstract algebra instructors attended to the students' mathematical thinking. Second, their responses nearly always aligned with their description and interpretation of this thinking. As such, differentiating based on these categories was not a meaningful way to distinguish the nature of the instructors' noticing. However, when going beyond connections to student work, the instructor responses ranged across our participants. In terms of interpretation of student work, we found that attention to formality was particularly helpful to distinguish amongst responses. Close to half of the instructors focused exclusively on elements of concept understanding (without attention to formality) while the other half of participants focused on aspects of formality. Definition was the most common formal aspect attended to. This is not particularly surprising in light of how closely formal definitions are tied to concepts at this level. Further, the student work responded to prompts that may engage them with definitions, but not require formal proofs. In terms of deciding how to respond, the most significant difference across our participants was the nature of the teacher moves. Guiding questions seemed particularly prevalent among our sample. This exploratory analysis of our data can support a more nuanced look at these responses. For example, to what degree are the guiding questions intended to funnel students towards correct answers versus lead towards open exploration? Due to our sample size, we are hesitant to make generalizability claims. Follow-up research may look explicitly at the role that specific tasks and experience play in this noticing through larger samples or qualitative interviews.

Implications of our study are mostly research-based. This exploratory study provided evidence that mathematicians' noticing (in terms of interpreting and responding) ranged greatly in even a small sample. Noticing at the tertiary level likely includes parallel constructs of noticing conceptual understanding and noticing formal representation aspects. Further, the nature of the participants responses illustrated components of the practice of responding to an individual student. The nature of these responses was significantly different than question types that were recently documented during lectures (Paoletti et al., 2018), and as such may serve as a starting ground to examine the instructor responses outside of the traditional lecture.

**References**


Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. *Research in collegiate mathematics education VII*, 63-92.

Boaler, J., & Humphreys, C. (2005). Connecting mathematical ideas: Middle school video cases to support teaching and learning. Portsmouth, NH: Heinemann.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational studies in Mathematics*, *27*(3), 267-305.

Herbel-Eisenmann, B., Drake, C., & Cirillo, M. (2009). "Muddying the clear waters": Teachers' take-up of the linguistic idea of revoicing. Teaching and Teacher Education, 25(2), 268–277.

Jacobs, V. R. & Spangler, D. A. (2017). Core practices in K-12 mathematics teaching. In J. Cai (Ed). Compendium of research in mathematics education (pp. 186-221). Reston, VA: National Council of Teachers of Mathematics.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education*, 169-202.

Johnson, E. (2013). Teachers' mathematical activity in inquiry-oriented instruction. *The Journal of Mathematical Behavior*, *32*(4), 761-775.

Johnson, E. M., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior*, *31*(1), 117-129.

Lajoie, C., & Mura, R. (2000). What's in a name? A learning difficulty in connection with cyclic groups. *For the learning of Mathematics*, *20*(3), 29-33.

Melhuish, K. M. (2015). The design and validation of a group theory concept inventory (Doctoral dissertation). Retrieved from http://pdxscholar.library.pdx.edu/open_access_etds/2490

Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education*, *49*(1), 9-38.

Milewski, A., & Strickland, S. (2016). (Toward) Developing a Common Language for Describing Instructional Practices of Responding: A Teacher-Generated Framework. *Mathematics Teacher Educator*, *4*(2), 126-144.

Moore, R. C. (2016). Mathematics professors' evaluation of students' proofs: A complex teaching practice. *International Journal of Research in Undergraduate Mathematics Education*, *2*(2), 246-278.

Nickerson, S. D., Lamb, L., & LaRochelle, R. (2017). Challenges in measuring secondary mathematics teachers' professional noticing of students' mathematical thinking. In *Teacher noticing: bridging and broadening perspectives, contexts, and frameworks* (pp. 381-398). Springer, Cham.

Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, *20*(2), 93-104.

Paoletti, T., Krupnik, V., Papadopoulos, D., Olsen, J., Fukawa-Connelly, T., & Weber, K. (2018). Teacher questioning and invitations to participate in advanced mathematics lectures. *Educational Studies in Mathematics*, *98*(1), 1-17.

Rasmussen, C., & Wawro, M. (2017). Post-calculus research in undergraduate mathematics education. In J. Cai, (Ed.), *Compendium for research in mathematics education* (pp. 551-579). Reston VA: National Council of Teachers of Mathematics.

Sahin, A., & Kulm, G. (2008). Sixth grade mathematics teachers' intentions and use of probing, guiding, and factual questions. *Journal of mathematics teacher education*, *11*(3), 221-241.

Sherin, M., & van Es, E. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of technology and teacher education*, *13*(3), 475-491.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher*, *15*(2), 4-14.

Simpson, A., & Haltiwanger, L. (2017). "This is the First Time I've Done This": Exploring secondary prospective mathematics teachers' noticing of students' mathematical thinking. *Journal of Mathematics Teacher Education*, *20*(4), 335-355.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*, 530-562.

Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*(2), 99-114.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of mathematics teacher education*, *11*(2), 107-125.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), 115-133.