"Zoom in Infinitely": Scaling-continuous Covariational Reasoning by Calculus Students

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Abstract: Recently, Ely & Ellis (2018) described a new mode of covariational reasoning scaling-continuous reasoning—and conjectured that it might support productive student thinking in calculus. We investigate that hypothesis by analyzing how calculus students employed scaling-continuous covariational reasoning when discussing differential calculus ideas. The interviewed students who took a course based on a "local straightness" approach to calculus used scaling-continuous reasoning in their description of the derivative at a point, particularly in their imagery of zooming in on a function at a point to reveal its slope. The interviewed students who took a course based on an "informal infinitesimals" approach to calculus used scaling-continuous reasoning in their account of how zooming in on a neighborhood reveals the coordination between a bit of x (dx) and the corresponding bit of y (dy), a relationship that gives a differential equation for that curve.

Keywords: covariation, differential, infinitesimal, local straightness, derivative

Ely and Ellis (2018) proposed the category of scaling-continuous variational/covariational reasoning and hypothesized ways it could productively support student reasoning in calculus. We build on this idea by investigating if and how scaling-continuous reasoning could support student understanding in single-variable differential calculus.

Theoretical Background

The idea of scaling-continuous reasoning is grounded in significant ongoing research on variational and covariational reasoning (e.g., Carlson, Jacobs, Coe, Larsen, and Hsu, 2002; Carlson, Persson, and Smith, 2003; Castillo-Garsow, 2012; 2013; Castillo-Garsow, Johnson, and Moore, 2013; Confrey & Smith, 1995; Saldanha & Thompson, 1998; Thompson, 1994; Thompson & Carlson, 2017; Thompson & Thompson, 1992). We briefly summarize several categories that are prominent in this research, as recently synthesized by Thompson and Carlson (2017). For a single quantity, *chunky-continuous variational reasoning* involves imagining that changes in a variable's values occurs only in completed iterated chunks, but without a clear image of how the variable actually takes on the intermediate values within each chunk. For two quantities, chunky-continuous covariational reasoning describes chunky reasoning with two quantities simultaneously: one quantity is taken in chunks, with corresponding chunks in the other quantity, but with no clear image of variation co-occurring within the chunks. Smoothcontinuous variational reasoning entails an image of a changing quantity that smoothly changes in time. The reasoning can imagine the variable's magnitude increasing in bits, but simultaneously anticipates smooth variation within each bit (Thompson & Carlson, 2017). Smooth-continuous covariational reasoning involves smooth variation in both quantities at the same time, including the understanding that smooth change in one quantity, no matter how small, can correspond to simultaneous smooth change in the other quantity. According to Thompson & Carlson (2017), smooth-continuous variational and covariational reasoning requires reasoning in terms of something moving in time. They describe smooth-continuous covariation essentially in terms of two quantities parametrized by an underlying time variable: "The coordination of quantities' values is like forming the pair [x(t), y(t)], where "t" stands for a value of conceptual time" (2017, pp. 444-5). Smooth-continuous reasoning has been shown to be robust and productive in calculus (e.g., Castillo-Garsow, 2012; Castillo-Garsow, Johnson, and Moore, 2013).

Scaling-continuous variational reasoning entails the image that at any scale the continuum remains continuous and that a variable takes on all of its values in that continuum. The continuum can be zoomed in on arbitrarily or even infinitely, and at no scale will it be revealed

as discrete or having holes. *Scaling-continuous covariational reasoning* involves imagining rescaling or zooming in on an increment of one variable quantity and coordinating that with an associated re-scaled increment of another variable quantity. For instance, one can envision shrinking or expanding a window of x-values and at every scale is a corresponding re-scaled window of y-values determined by the correspondence between increments of x and y. Unlike smooth-continuous reasoning, this does not fundamentally rely on an image of motion or an underlying time parameter. Scaling-continuous reasoning itself entails the idea that it is possible to zoom arbitrarily to any (finite) scale, but it plausibly requires another mental act to generalize or encapsulate this to develop an image of zooming in *infinitely*, revealing infinitesimal increments. We also note that scaling-continuous reasoning does not by itself entail the ability to effectively calculate at any scale (just as smooth-continuous reasoning does not alone entail the ability to effectively calculate change in one quantity in terms of change in another).

Method

Each author taught a calculus class using different non-traditional approaches—local straightness (Samuels) and informal infinitesimals (Ely)—conducting various semi-structured interviews investigating the reasoning of students in the classes. For this study, we analyzed these interviews with an eye to how different types of covariational reasoning manifested.

Setting 1: A Calc I class with a local straightness approach

Author 2 (Samuels) taught a Calculus I class using local straightness as a cognitive root (Tall & McGowen & DeMarois, 2000) for the derivative and the integral. Local straightness is the property that zooming in at one point on the graph of a function of one variable reveals a (nearly) straight line when the function is differentiable at that point, and the slope of the line is the derivative at that point (Samuels, 2017). Student-centered guided discovery activities were at the core of the curricular design.

Students first developed the idea of the derivative at a point by engaging in activities using an applet with two windows. One window contains the graph of the function on a fixed scale. The second window graphs the function centered at a variable point on the graph on a variable scale. (The point and the scale can each be manipulated by the user with sliders; A box in the first window indicates which portion of the graph appears in the second.) After zooming in, students see a (mostly) straight line, and learn to associate the slope of that line with the slope or derivative at that point. (If the function is not differentiable at that point, a straight line never comes into view.) Questions and activities for the students included: describing what is visible during the zooming process, estimating slope at a point, and making a table of slope values. For a more detailed description of the approach, see (Samuels, 2017). Algebraic limits and their application to the slope difference quotient typically are presented as an entrée to the derivative (e.g. Stewart, 2012) and are seen as a necessary precursor to understanding the derivative (Zandieh, 2000); in this curriculum, they are reserved until the end of the course. The geometry of local straightness replaces the symbolic formalism of the limit definition as a way to conceive of the derivative. Further, in this approach, the slope object is not an encapsulation of a limit process, as it is when you move the second point along the graph toward a fixed point and secant lines must be understood to approach a tangent line. In that process, secant lines are first constructed as additional mathematical objects. Instead, the local slope is in some sense already there to be "found" for the student; once one zooms in close enough one can see the graph as being straight (enough) and thus having a slope. Here, no additional mathematical objects are constructed; rather, we take a different view of the existing graph.

Setting 2: A Calc I class with an informal infinitesimals approach

In Fall 2016, the first author (Ely) taught a Calculus I course that used an "informal infinitesimals" approach in a large lecture (110 students). His purpose was to build calculus ideas in such a way that the notation transparently referred to quantities, rather than serving as a shorthand for the result of a limit process. This is in keeping with the imagery Leibniz had in

mind when developing the notation we still use for calculus: dx denotes an infinitesimal amount of x and \int represents a sum of infinitely many infinitesimal bits. The class used Leibniz' heuristics for imagining infinitesimals, and his consistent rules for working with them. The purpose was to allow students to work directly with infinitesimal quantities using regular arithmetic and algebraic operations. For instance, dy/dx was a quotient of two infinitesimal quantities, not code language for $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Although the development of the hyperreal numbers in the 1960s offers a formal system sufficient for rigorously grounding Leibniz' approach (Robinson, 1961), the informal infinitesimals calculus class used Leibniz' notation and imagery with very limited reference to the formal hyperreal numbers. For a detailed summary of how infinitesimals can be rigorously developed in this manner, see the appendix of (Ely, 2017).

An infinitesimal is a number or quantity smaller than any real number but larger than 0. In lecture, the instructor used the image of an infinitesimal distance as being revealed by zooming in infinitely on the real number line. For instance, if you zoom infinitely on the point 100 on the real number line, using an infinitesimal scale factor of ε :1, you can see a little neighborhood or "monad" around 100 that contains an entire world of numbers that are all infinitely close to 100, including numbers such as $100 + \varepsilon$ and $100 - 3\varepsilon$. Infinitesimals such as $100 - 4\varepsilon$, are still indistinguishable from 100 at this zoom factor of ε ; these are revealed by zooming *again* infinitely at 100 by another scale factor of ε :1, and thus are considered second-order with respect to the infinitesimal ε . This image can be formalized in the hyperreal numbers (e.g. Keisler, 1986), although the informal infinitesimals class did not do so.

Focusing just on differential calculus, the class developed methods for deriving differential or "bits" equations from "amounts" equations. For example, y = x was seen as an equation that gives an amount y in terms of an amount x; its bits equation $dy = 2x \cdot dx$ provides a bit of y (dy) in terms of a bit of x (dx), which relies on the value x near which the variation is occurring. Later, after working with bits equations, we divided both sides of a bits equation by an infinitesimal to find the quotient of two bits near a particular x: i.e., dy/dx = 2x. In the special case where y is a function of x in the original amounts function, this quotient will also be a function of x, which enables the defining of that amounts function's derivative function. Bits equations were also used extensively in the course as a basis for definite and indefinite integrals, a development that is beyond the scope of this article (but see Ely 2017 for more detail).

Data collection

Both authors conducted semi-structured clinical interviews with students about a variety of topics from a Calculus I class they were just completing, 7 students that used the informal infinitesimals approach and 25 that used the local straightness approach. Interviews were analyzed for student use of various types of covariational reasoning. For this paper, we focus on how scaling-continuous reasoning manifested and supported student understanding and explanation of several important ideas in differential calculus.

Results

Scaling-continuous student reasoning in local straightness calculus class

Multiple students discussed the derivative using scaling-continuous reasoning. This occurred both in their general conception and in solving specific problems. For brevity, here we relate three excerpts, each with illustrative verbal and graphical components.

The interviewer asked Young to describe his process of determining the derivative of a function at a particular point x. He said that "[while zooming in] the line straightens out in the zoom window." Subsequently, to explain this process, he drew the picture in Figure 2a.

He indicated his focus on a single point with a black dot on the graph. He indicated his zoom action by drawing, first, a box around this point, and second, that box magnified. (The paper was rotated during the discussion.) Young's work suggests he is using scaling-continuous covariational reasoning. He zooms, then draws a re-scaled window to show the imagined result

of the zooming action. In the magnified image, the zoomed-in neighborhood on the graph is represented as continuous, unbroken, and (essentially) straight. This straightness allows him to coordinate the vertical and horizontal variation in order to find a slope of the graph in that neighborhood.



To explain the derivative at a point, Carl drew a graph with a tangent line at one point. He then elaborated, "To get this tangent line, we learned from the lab that it could be there, and there (draws 3 lines going from curvy to straight, in Figure 2b), you zoom in enough, and it becomes a straight line. It's got to become a straight line or you don't have a derivative." He focused on a unique point, and the nature of the function at multiple levels of zooming, a strong indication of scaling-continuous reasoning.

A third student, Sam, also used scaling-continuous covariational reasoning in his description of the derivative at a point and how it can be calculated. He goes further than the other two students in that he also distinguishes between zooming arbitrarily to get an approximate value and zooming infinitely to get an exact one:

- J: What is a derivative?
- Sam: Derivative is slope at a point. That's the bottom line. ... If the graph is like this (draws image in Figure 2c), the derivative, as you zoom in, this is the tangent line. The derivative becomes more and more accurate.
- J: So you also mentioned the tangent line. What does the tangent line have to do with the derivative?
- Sam: The tangent line is the slope at a point. As the tangent line moves this way (gesturing to the right), it gets more and more steep. So that's the derivative.
- J: When you find the derivative, when you give an answer, is it approximate or exact?
- Sam: It's approximate.
- J: Is there an exact answer?
- Sam: If you zoom in infinitely. It's not perfectly accurate. The main concept of finding the derivative, I think, is seeing this curve as a collection of straight lines. But it's not really a collection of straight lines, it's a curve. And the straight lines are the tangent lines.

Sam's account of slope at a point uses scaling-continuous covariational reasoning in several ways. He indicates his focus on a single point with no secondary point with a single black dot on the graph. Like Young, he describes "zooming in" to find a derivative at that point, suggesting that each zoom entails a coordinated horizontal and vertical re-scaling. With each zoom, the derivative becomes more accurate, but it is still "approximate." This indicates he is picturing scaling revealing covariation at an arbitrary level. Then he explicitly adds that one can "zoom in infinitely" to get an exact answer. His description indicates that he is generalizing his image of arbitrary re-scaling: at the infinitesimal scale there is still smooth covariation, and the graph has become perfectly straight, enabling the determination of an exact slope. His "collection of straight lines" metaphor is a way to hold both finite and infinite scaling conceptions; it was, in fact, also used by Leibniz (Katz, 1998).

Scaling-continuous student reasoning in informal infinitesimals differential calculus

Several of the students who took the informal infinitesimals calculus class employed scalingcontinuous covariational reasoning when interviewed in their reasoning with differential notation and differential equations. For sake of brevity, we describe this with an illustrative segment of one interview. In this segment, the interviewer has asked the student, Roan, to describe the relationship between the amounts equation y = x and its corresponding bits equation $dy = 2x \cdot dx$. The interviewer asks what the terms in the bits equation mean. Roan describes how the dx refers to an infinitesimal difference between two x values, and the dy refers to an infinitesimal increment between the two corresponding y values. The interviewer then asks what the x is doing in the equation. After some discussion, Roan asks if he can illustrate his thinking with the dynamic graphing program Desmos on his computer. He graphs the function y = x and then says that the bits equation $dy = 2x \cdot dx$ needs an x in it because for this function the dy's will be different sizes depending on the dx's. The interviewer then asks him to explain his thinking in terms of dx and dy increments.

Roan's computer has a touch screen which enables him to zoom in and out on the graph in the Desmos program by using two fingers. He zooms in on the graph at the origin, and points out that near 0 "the proportion to dy to dx is not much at all," gesturing a vertical increment (dy) that is small in comparison with the horizontal increment (dx). Roan then zooms back out and says:

Roan's words	Roan's gestures
Yeah, 'cause you can see, like, as	gestures with two fingers significantly separated
you go across this distance,	
y doesn't change as much as here,	gestures with two fingers close together a dx
like if you go from here to here,	increment in one place and then another same-sized
	dx increment further to the right
y goes up more in relation.	gestures with the corresponding vertical dy
	increments of two different sizes, the right one
	being significantly larger than the left one
Or from here to here,	zooms in
it goes up this much, so it's going	gestures a fixed small horizontal increment from x
up more and more in comparison.	= 0 to 0.2 and then again from $x = 0.2$ to 0.4, then a
	few more times, moving the increment to the right
So the change isn't affecting y as	zooms out, then drags the graph over and indicates
much and then you keep going	a small dx increment in a different spot
over. Now when x changes,	
<i>y</i> goes a lot.	gestures a vertical increment
Then when you keep going over,	drags graph over and indicates another same-sized
	small dx increment yet another spot further right
when you change your <i>x</i> , <i>y</i>	gestures a large vertical increment
changes a lot.	

Roan then zooms out further. The interviewer asks about how this relates to the x, and Roan says, while pointing at the indicated parts of the equation $dy = 2x \cdot dx$, "Because this [dx] stays the same, and this [x or maybe 2x], is giving the proportion, where this [dx] is fixed..." He describes then how as you move to the right, x gets larger, and the dy increment gets larger even though the dx stays the same.

In this segment, Roan treats the increments dx and dy as small differences in the variable quantities x and y in the graph of y = x. He describes how the bits equation dy = 2x dx shows the coordination of uniform-sized dx increments with varying-sized dy increments, and that this variation depends on where in the x direction the increments are being considered.

Roan's continual gesturing shows how scaling-continuous covariational reasoning supports his understanding of this coordination between dx and dy. In two minutes, Roan zooms in or out on the graph no fewer than twelve times. He zooms in on the graph usually when he is talking about a particular increment dx and its corresponding dy. This suggests that his image is that an "infinitesimal" (as he often calls it) difference or increment is obtained by zooming in near some point x. When it comes time to talk about how a dx-dy pair at one spot x_1 relates to another dx-dypair at another spot x_2 , he zooms back out again so that the overall shape of the graph is more apparent, gesturing how the dy's are different in size at these two locations. Scaling in is part of his image of how one sees a pair of infinitesimal increments in the two coordinated variables at a particular location. Scaling out is part of his image of how the coordination between the dx and dy itself varies from point to point on the larger graph.

In his image, there seems to be an operational coordination between increments of x and increments of y at every scale, which also presumes that scaling never reveals non-intervals in either quantity. Because this coordination is available even, according to Roan, at the infinitesimal scale, he can envision a distinct coordination of dy and dx "at every x."

Discussion

Neither calculus course was designed or taught with the idea of scaling-continuous variational/covariational reasoning in mind—indeed, at the time neither instructor had heard of the idea. Yet some of the students in the courses ended up displaying these modes of reasoning, and these modes seem to support these students' reasoning about some key ideas in differential calculus. In this section we discuss how scaling-continuous reasoning can be seen to support robust understandings of some key ideas in differential calculus that are aligned with the goals of the two classes.

Students in the local straightness calculus class frequently exhibited scaling-continuous covariational reasoning when discussing a derivative at a point. They anchored focus at a single point, which they indicated both verbally and with a graphical mark, and pictured zooming in as far as needed, with a technology tool or with mental or written images, to reveal a straight line segment. They then estimated the value of the slope and assigned it the meaning of the derivative of the original function at that point. In this last step, they turned to coordinating increments in both quantities at a single point, recognizing that the arbitrary zooming of scaling-continuous reasoning was necessary to make that meaningful.

Also, it is notable that this can serve as a foundation for the conception of the derivative as a function, as demonstrated by Sam (and by many students in class). He described taking the straight line at a point and moving it to the right and recording the derivative at every point. This indicates he had encapsulated his scaling-continuous construction of the tangent line, to recreate it at any point.

In the informal infinitesimals calculus class, scaling-continuous variational reasoning provides a crucial image that at each scale the values of a continuous variable form a continuous unbroken increment on which variation occurs. This idea can then be generalized to an image that each infinitesimal increment looks the same way, a generalization that Sam and Roan both appear to have made. A robust image of infinitesimal entails generalizing or encapsulating the process of scaling involved in applying scaling-continuous variational reasoning.

With this in mind, scaling-continuous covariational reasoning gives the student a way to imagine a coordination between each continuous increment of one variable and a continuous increment of another, at every scale. Roan tacitly assumes that coordination when gesturing and speaking about the relationship between bits, differences, and changes in x and y. While scaling-continuous covariational reasoning only includes this coordination for arbitrary scales, for the informal infinitesimals approach it is important for this coordination at some point to be generalized to the infinitesimal scale. The reason is that this provides a basis for the productive interpretation of a bits (differential) equation as an algebraic description of the relationship between an infinitesimal amount of change in, say, x and a corresponding infinitesimal amount of change in, say, y. Because these amounts are infinitesimal, this coordination can be envisioned

at every value of x, and depending on that value of x. This is illustrated when Roan describes and gestures how the coordination he imagines between dy and dx is established at different points, and how this in turn varies from location to location.

In both classes, the encapsulation of scaling-continuous covariational reasoning at a single point is a crucial element as students form their conceptions of single variable differential calculus, even though it manifests differently.

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