What is a Differential? Ask Seven Mathematicians, Get Seven Different Answers

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The symbol "dx" is one example of a differential, which is a calculus symbol that is found in a variety of settings and expressions. We wanted to explore how expert mathematicians think about differentials in some of these settings and expressions, in order to see what levels of consistency might appear among their views. To that end, we created an interview protocol that contained differentials in the contexts of derivatives, definite and indefinite integrals, and separable differential equations, interviewed seven mathematicians, and analyzed their responses using a form of thematic analysis. Overall, we found no instances of total agreement among all subjects, but did find several common and recurring themes, including some that were unexpected and not found in our previous studies.

Keywords: Differentials, Calculus, Concept Image, Derivatives, Integrals

In this contributed report, we analyze how seven mathematicians view the roles, if any, that differentials play within various mathematical expressions and situations. When discussing the term "differential," we refer to a letter d followed by a second letter that is usually dependent on a particular context. Examples of these include dx, dt, and dA, and for this paper, we will use "dx" to reference a generic differential. These symbols are common in calculus, and can be found in many places, including Leibniz notation for derivatives, definite and indefinite integrals, the process of integration by substitution, and several types of differential equations.

We have found research in both mathematics and physics education literature that describes how students perceive the dx in a definite integral. For some students, this differential might have no meaning at all (Artigue, 1991; Hu & Rebello, 2013). If it does have a meaning, it might only serve to indicate the variable of integration (Artigue, 1991; Jones, 2015), or it could represent a small amount of a quantity (Artigue, 1991; Nguyen & Rebello, 2011; Von Korff & Rebello, 2012) or a small change in a quantity (Sealey & Thompson, 2016; Von Korff & Rebello, 2012). Outside of these particular student interpretations, a differential might function as a linear estimate (Henry, 2010; López-Gay, Martinez, & Martinez, 2015) or represent a formally-defined infinitesimal as found in nonstandard analysis (Keisler, 2012; Robinson, 1961).

Most of this particular literature discusses student interpretations of the definite integral, but only minimally addresses the interpretations of the instructors and expert mathematicians who teach these students. We have felt that there is an opportunity to broaden the above research by expanding the list of expressions containing differentials as well as exploring the interpretations of experienced mathematicians. Therefore, the main research question we address in this paper is "What concept image(s) (Tall & Vinner, 1981) do expert mathematicians hold of the differential throughout its various mathematical contexts?" Two other areas we wish to explore are analyzing each expert's interviews to see how consistent his or her responses are throughout the interview, and looking at each context in which a differential exists (e.g. indefinite integrals) and comparing each expert's views on the differentials in that context, to see what patterns or consistencies, if any, might emerge.

Preliminary work was conducted via two smaller-scale studies. An initial study involved four mathematicians who were asked about how they conceived the differentials in expressions involving integration, Leibniz derivative notation, integration by substitution, and ordinary differential equations. We concluded that, while some subjects gave common responses at times, there was no overarching formal concept definition for the differential (McCarty & Sealey, 2017). A second study included two mathematicians and one physicist who were interviewed about similar expressions and contexts, and found not only a similar lack of an overall formal concept definition for differentials, but also the suggestions of a split between mathematicians' views and physicists' views. (McCarty & Sealey, 2018). In this current paper, we focus only on mathematician interviews and leave physicist interviews for future research.

Theoretical Perspective

Discussing the notations $\frac{dy}{dx}$ and $\int f(x)\,dx$, Tall (1993) questions what relationship might exist between the two "dx" portions of those notations and notes:

> Giving a modern meaning to these terms that allows a consistent meaningful interpretation for all contexts in the calculus is possible but not universally recognized. On the other hand, failing to give a satisfactory coherent meaning leads to cognitive conflict which is usually resolved by keeping the various meanings of the differential in separate compartments. (Tall, 1993, p. 6)

Thus, one might use only one conceptualization for all differentials at all times, or one might possess and use different conceptualizations for differentials depending upon the context in which they are found (for example, viewing the dx in an indefinite integral as indicating the variable of integration and the dx in the derivative notation $\frac{dy}{dx}$ as a small amount of the quantity represented by the independent variable, x.)

Because multiple interpretations of differentials are possible, we believe that Tall and Vinner's (1981) *concept image* is an appropriate theoretical perspective for our research. Concept image is defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p.152), and if one has multiple interpretations of differentials, then the words "total" and "all" in that quote take on greater meaning. During our interviews, we attempted to gain as complete an understanding of our subjects' concept images as possible, with the following questions in mind: Within these possible multiple interpretations, would any subjects exhibit *potential conflict factors*, defined as aspects of their concept image that showed contradiction? If so, would they be aware of any of their contradictions, making them *cognitive conflict factors*? Would all subjects' responses be able to be distilled into a *personal concept definition* that fully defined how they viewed differentials, and if so, would multiple personal concept definitions be able to come together to form a possible *formal concept definition*?

Methods

For this study, seven mathematicians (pseudonyms André, Bryan, Christopher, Diane, Eugene, Francis, and Gustav) from the same large research university were given semi-structured interviews that used the interview protocol summarized in Table 1. Each subject was asked the same questions about the expressions and contexts given in the protocol, but follow-up questions were asked when needed to clarify subjects' initial responses. Including these additional questions and introductory questions that asked the subjects' background information, the average length of the interviews was approximately forty-five minutes. All interviews were videorecorded, with six interviews conducted in person, and a seventh conducted over Skype and recorded with Open Broadcasting Software.

Data analysis was done in the style of Braun and Clarke's (2006) thematic analysis. The videotaped interviews were transcribed and analyzed for data points, which we defined to be the specific instances in which differentials were discussed. These data points were assigned codes based on how we perceived the tenor of the subjects' views toward the differentials. The lists of codes from all seven interviews were analyzed, and similar codes found across multiple interviews were pulled together, to create an initial list of themes. The themes in this initial list were compared with one another to see which of them might be consolidated and streamlined into a smaller list of larger, overarching themes. Finally, the transcriptions were read one last time and compared with this final list of themes, to make sure that the themes described by this list encompassed all responses within the entire data set.

Table 1
A Summary of Our Interview Protocol

Description	The Specific Questions
Five Expressions Presented with no Context	• $\frac{dy}{dx}$, $\int_a^b f(x)\,dx$, $\int g(x)\,dx$, $\int_0^1\int_2^3 f(x,y)\,dy\,dx$, and $dy = 2x\,dx$ • For each of these, subjects were asked how they conceptualized the differentials in the expressions, and whether they thought the differentials had (a) a graphical representation, and (b) a size.
Three Expressions Presented within a Context	• A "Law of Cooling" ODE: $\frac{d\tau}{dt} = -k\tau,\ \ \tau(0) = 20$ • A "Work" problem involving the integral $\int_0^{50} 700 - 3x\,dx$ • $du = \frac{1}{2\sqrt{t}}\,dt$, used in the evaluation of the integral $\int_1^4 \frac{\cos\sqrt{t}}{2\sqrt{t}}\,dt$
Three Additional Questions	• At the beginning of the interview, subjects were asked what the word "differential" meant to them. • After the word "Delta" was first mentioned by the subject, he or she was asked to clarify the differences, if any, between Δx and dx. • After their first use of a phrase like "infinitely/infinitesimally small," subjects were asked if they could clarify/quantify their phrase.

Data and Results

We found many themes during data analysis, some of which were expected from our prior research and our analysis of recent literature, some that were new to us, and some that were stronger than expected. We summarize the major themes below.

Algebra with Differentials versus "Algebra" with Differentials

The use of the quotation marks in this subtitle is to represent the idea that some experts were not willing to describe certain common manipulations of differentials by directly using words like "multiply," "divide," and/or "cancel." To give one example, when some subjects brought up "Chain Rule" notation, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, Bryan, Christopher, and Diane each had no problem with notating it this way, but stopped short at saying that what was happening was true division or cancelling of the dx. Christopher said that it was "as if" we cancelled the dx, Bryan said that "there's a little bit more going on than just cancelling," and Diane said that she wasn't sure if they cancel, and that books "come up with some funny, hand-wavy thing to explain what they're doing there."

Another example of "algebra" with differentials occurred during the discussion of the separable ODE $\frac{d\tau}{dt} = -k\tau$. Subjects who claimed either that the expression $\frac{d\tau}{dt}$ was not a ratio (Eugene, Francis) or that they weren't sure if it was a ratio (Diane) still ended up separating the expression when solving the ODE. This separation was rationalized by either claiming that this separation stood in for the integration $\int \frac{d\tau}{dt}\, dt = \int -kt\, dt$ (Eugene, Francis), or that we just "think of" dt as being a quantity and act like we're "multiplying" (Diane). It is perhaps worth noting that, even though some subjects refused to say personally that separation of variables entailed "multiplying by dt," none of the subjects would outright object if their students described their solution to a separation of variables problem this way. Five of the seven subjects said they would have no problem if their students used the words "multiply by dt," while the other two (Diane and Eugene) were not certain if they would allow their students to do this.

There were clearer statements of actual algebra made as well. Some subjects stated directly that one could manipulate differentials by multiplying or dividing, and there were statements that implied multiplication and division were acceptable, including André's and Christopher's separation of variables in the ODE without any qualms as to the legality of such multiplication. There were also contrary, clear statements that one could not multiply nor divide, and some of these "Yes, you can" and "No, you can't" statements were in direct opposition to one another. One example of this was Bryan and Christopher saying that the "$f(x)dx$" in $\int_a^b f(x)\, dx$ was an actual multiplication of $f(x)$ and dx and Diane saying that it was not a multiplication.

Subjects' Uneasiness with Differentials

Given the lack of consensus found in all of our studies and the lack of a clear formal concept definition for differentials, it was not surprising that some subjects admitted a level of uncertainty to some of their responses. This uncertainty manifested itself in various ways: some subjects claimed that they had no formal definition for some of our expressions, some claimed that they had an intuition about the expressions but could not put this intuition into words, and some gave a partial explanation while admitting that they knew there was "more" to the concept but that they could not put this "more" into words.

There were definite instances of cognitive conflict factors. To give one example, Francis noted and called attention to his conflicting statements when they occurred. After claiming the differentials in the earlier expressions $\frac{dy}{dx}$, $\int_a^b f(x)\, dx$, $\int g(x)\, dx$, and $\int_0^1 \int_2^3 f(x,y)\, dy\, dx$ had no size, he gave what we call the standard "linear approximation" explanation of $dy = 2x\, dx$, stating that these dy and dx were measurable quantities. He noted the inconsistency, saying "… now I'm being cognizant of what I think about this, and what I originally said, no. That these [pointing at the dy and dx] are <u>not</u> quantifiable. [Thinking] And I'd have to really think about rectifying this."

The dx is a Real Number or a Formal Infinitesimal

Our previous work as well as the recent literature shows that an interpretation of a dx as an unquantified, not formally-defined "small" amount is common and not unexpected; what was slightly unexpected in our research was the emergence of themes in which subjects specifically stated that the dx represented a real number or a formal infinitesimal. Francis mentioned one area in which textbooks commonly assert that dx and dy are real, the idea of linear approximation, usually represented as $\Delta y \approx dy = f'(x)dx$. André and Bryan described some dx as being on a smaller scale than every other entity in the problem, a description I liken to Courant and John's (1965) "physically infinitesimal." For example, Bryan defined his dx as "relatively small," and gave examples of dx possibly equaling 100,000 miles if one is discussing astronomical phenomena, but dx equaling one Ångström if one is discussing molecules. Either way, no matter at what scale one is measuring a specific problem, for these two subjects, the dx represents a real number. For the purposes of this report, we can define a positive, nonstandard analysis infinitesimal, ϵ, as $0 < \epsilon < r$ where r is any real number (Keisler, 2012). Gustav directly stated that one could view any dx as one of these formal infinitesimals, and while Eugene and Francis did not view differentials in this way, they acknowledged that others might, and that formal infinitesimals were a valid interpretation of differentials.

The dx is not Specifically Sized

This is a common theme, found both in the literature and in our previous work, though our current research has found more nuance to this theme than we reported previously (McCarty & Sealey, 2017). A differential might be described or implied to be "small" without a precise definition of what "small" means (as opposed to defining differentials as real numbers or formal infinitesimals, both concepts with precise definitions.) This occurred at the beginning of every interview, when the subjects were asked what the word "differential" meant to them, and all replies contained a reference to "smallness" that was not explicitly defined. Other versions of this were Diane describing differentials as "infinitely small" while claiming that "infinitely small" could not be defined, and Eugene claiming that the dx was a small entity that was the result of the limit of Δx going to zero.

We include in this theme comments that did not state directly but seemed to imply that the dx might be a real number or formal infinitesimal. Eugene described the dx in a definite integral as being a stage in the limit process. In this case, if Δx is going to zero step-by-step, and the dx represents one of those steps, must not dx be a real number? Other subjects made statements that might be interpreted as referencing nonstandard infinitesimals. André described the dx as being "what's left of Δx after it goes to zero," and Diane said that when the two points that define a secant line "are on top of each other", then we can think of the Δx as a dx. One might interpret both of these ideas in a nonstandard manner: in each case, the subject describes a process that goes through all real numbers and results in a distance of zero, yet the dx still exists. This might be possible if one views these dx as the epsilon described above: an entity that still exists yet is outside of the reals.

The dx Indicates a Variable or Process

It is also possible that a differential might not have a size because it indicates a variable or references a process. Differentials might only be used to call attention to a particular variable, as in the dx serving as an indicator of the variable of integration in an indefinite integral or the dy and dx indicating the "directions" of integration in the double integral. Differentials might also serve to indicate a process, with some subjects saying that the dx in a definite integral only represented that the limit of a Riemann sum was taken, and that a "u-substitution" made in the evaluation of an integral was a representation of the Chain Rule.

A small sample of the themes we found in the discussions of some of our questions can be found in Table 2. A quick look at this table and the number of themes found in it can determine our answers to the questions posed earlier in this paper: there is no formal concept image for the differential across all contexts, and only some areas of consistency within one expert or within one expression. Many experts stated that one's views on differentials also depends on the context in which the differentials were presented, and thus it is even possible to discuss inconsistencies at a level more fine than the level implied by this table.

Table 2
A Summary of Some of Our Results (Only the Expressions Presented without Context)

Expression	André	Bryan	Chris	Diane	Eugene	Francis	Gustav
$\frac{dy}{dx}$	A, C, R	C, R, V	IR, P	P, S	V	II, P, V	I, V
$\int_a^b f(x)\,dx$	II, P	A, R, P	P, S	P, S, V	C, S, V	P	C, I, V
$\int g(x)\,dx$	V	N	A, S	V	V	N	V
$\int_0^1 \int_2^3 f(x,y)\,dy\,dx$	C, II, P, V	C, P, R	S	P	P, S, V	C, P, S, V	V
$dy = 2x\,dx$	P	C, U, R	A, IR, S	"A", U	P, U	R, U	A, R, S

The letters in the table correspond to the presence of the themes described above:
A: Algebra with Differentials, "A": "Algebra" with Differentials, C: Differential Interpretation Depends on Context, I: Differential is a Formal Infinitesimal, II: Differential is an Implied Infinitesimal, IR: Differential is an Implied Real Number, N: Differential Has No Meaning, P: Differential Represents a Process, R: Differential is a Real Number, S: Differential is "Small" (Not Specifically Sized), U: Subject Expressed Uneasiness about Differentials, V: Differential Indicates a Variable

Discussion

Given the number of themes we found in our research and the number of different opinions within each theme, it should not be surprising that we conclude there is no formal concept definition of the differential. To be more direct, we found no instances where all seven subjects agreed on the interpretation of any one differential in any one mathematical context. It appears that the second half of Tall's (1983) quote applies, and that the lack of one overarching meaning for dx means that our subjects' concept images of dx consist of many different meanings for the differential compartmentalized in separate "locations."

This leads to some implications for instruction and suggestions for future research. One might ask if it matters that individuals possess such disparate views of the differential. After all, these seven subjects are accomplished mathematicians and experienced instructors; the fact that each of them views differentials in their own way did not prevent them from earning their doctorates. However, one might counter that argument with the idea that many, if not most, notations in mathematics are not ambiguous at all. For example, we would submit that a study that asked subjects their interpretations of the notations "Σ", "$\sqrt[3]{\ }$", and "$!$" would show no ambiguity in subject responses. If many notations have only one clear, direct, single interpretation, one might argue that dx should have one clear, direct, single interpretation as well. Indeed, a few subjects in our study expressed personal discomfort when noting that sometimes differentials are taught in a "hand-wavy" way, without real support (Diane), or that instructors sometimes teach differentials less formally than they should (Francis). We suggest that the reason for this discomfort is the fact that there is no consensus on what a differential is. Perhaps further research could investigate how (or if) instructors having disparate views of the differential affects student learning.

Another teaching implication might come from the first past of Tall's (1983) quote: "Giving a modern meaning to these terms that allows a consistent meaningful interpretation for all contexts in the calculus is possible but not universally recognized. (p.6)" It is possible that the differential as a nonstandard analysis infinitesimal would be the most consistent approach. There are certainly textbooks that teach calculus this way (e.g. Henle & Kleinberg, 2003; Keisler, 2012), but, as Tall stated, such an approach is not universally recognized. Further research might explore the efficacy of such an approach. An idea for future research comes from the notion, mentioned above, that some subjects claimed that there were "Physics" and "Mathematics" approaches to differentials. This idea was touched upon in our pilot study (McCarty & Sealey, 2018) but not in this study. Further research might wish to explore how physicists view differentials and how consistent their views are with mathematicians', especially since many first-year physics majors take calculus classes that are taught by mathematicians.

We conclude this paper by quoting what Christopher said at the end of his interview, regarding the usefulness of differentials:

> Yeah, they're very useful, 'cause they have a lot of content. There's a lot of, sort of conceptual content in there, and if you shy away from them, you're robbing the students of sort of conceptual content where they can think about things – these things actually mean something, rather than being things that are so abstruse that they can only be handled with a course in advanced calculus. I think a lot of that – all that developed just from physical reasoning and – although the mathematics by itself is not rigorous, you can make it rigorous, and the reasoning is valid. So I don't see any reason to avoid talking about them

At this time, we are in no position to say with certainty that one view of differentials is superior to any other. If there were any conclusion we might make, it is that we are in agreement with our interview subjects who are not comfortable with textbooks or teaching methods that either ignore differentials entirely or give them short shrift. We agree with Christopher that differentials are useful and worthy of classroom discussion, and it is our hope that our research inspires and motivates further work that will help explore the utility of differentials.

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