Reasoning Covariationally to Distinguish between Quadratic and Exponential Growth

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In this report, we present preliminary findings from clinical interviews examining inservice teachers' understandings of quadratic growth and exponential growth. The purpose of this pilot study is to investigate how teachers may naturally leverage covariational reasoning to distinguish between the two types of growth. In this report, we first present relevant constructs pertaining to teachers' covariational reasoning and then describe one task we used in clinical interviews. We then present preliminary findings regarding how teachers' leveraged (or did not leverage) covariational reasoning as they addressed this task to differentiate between quadratic and exponential growth. We conclude with preliminary implications and questions regarding how these preliminary findings may have implications for a larger study with pre-service secondary mathematics teachers.

Keywords: Covariational reasoning, quadratic growth, exponential growth

Although several studies have indicated that covariational reasoning can support students develop various mathematical ideas, such as quadratic, exponential, trigonometric, and parametric functions (e.g., Castillo-Garsow, 2012; Ellis & Grinstead, 2008; Johnson, 2012; Moore, 2014; Paoletti & Moore, 2017) the available research suggests that reasoning covariationally is uncommon and minimal among high school mathematics teachers in the U.S. (Strom, 2006; Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017). Particular to quadratic and exponential growth, researchers have indiciated middle school students (e.g., Ellis, 2011a; Ellis, Özgür, Kulow, Williams, & Amidon, 2015) can reason covariationally to construct and reason about quadratic and exponential relationships, but there is limited research examining pre-service and in-service teachers' understandings of these growth patterns. Consequently, the aim of this pilot study was to examine teachers' meanings related to quadratic growth and exponential growth. We present preliminary findings from clinical interviews providing insights we will leverage in implemeting a semester long teaching experiment with undergraduate pre-service teachers. We address the research question: "How might teachers reason covariationally to differentiate between quadratic growth and exponential growth?

Theoretical Perspective

Researchers have articulated varied perspectives of covariational reasoning. Providing a contrast to an emphasis on functions as representing correspondence rules, Confrey and Smith (1994) advocated a covariational approach to function that involves coordinating successive values of one variable (y_m to y_{m+1}) with successive values of another variable (x_m to x_{m+1}). Whereas Confrey and Smith focused on coordinating sequences of numeric values, Saldanha and Thompson (1998) proposed a more continuous perspective on covariational reasoning as "someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). This involves the student imagining both quantities being tracked for some duration and understanding that "if either quantity has different values at different times, it changed from one to another by assuming all intermediate values" (p. 298). For example, as the side length of a square increases continuously from 4 units to 5 units, taking all intermediate values between 4 and 5 units, the area increases continuously from 16 sq. units to 25 sq. units, taking all intermediate values between 16 and 25 sq. units.

Building on these and other researchers' characterizations, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) proposed a framework encompassing five mental actions that students engage in when reasoning covariationally. The mental actions involve identifying *change in two quantities* (MA1), the *direction of change* of one quantity with respect to the second quantity (MA2), *amounts of change in one quantity for equal changes in the second quantity* (MA 3), and the *average and instantaneous rate of change* of one quantity with respect to the second quantity (MA 4-5).

Paoletti & Moore (2017) noted that leveraging these different forms of covariational reasoning can support students in developing more robust quantitative structures and a better understanding of the relationships they are representing. In this report, we focus on inservice teachers' covariational reasoning as they conceived of and described quadratic and exponential growth.

Literature Review: Conceptualizing Quadratic and Exponential Growth

Although there are studies (e.g., Chazan, 2006; Zaslavsky, 1997) pointing to student misconceptions related to quadratic and exponential growth, there are fewer studies providing evidence of students or teachers maintaining productive understandings of these ideas. We briefly describe researchers' characterizations of productive understandings of quadratic and exponential growth using a covariational reasoning lens and synthesize key findings from these studies.

With respect to quadratic growth, researchers (Ellis, 2011b; Lobato, Hohensee, Rhodehamel, & Diamond, 2012) taking a covariational lens compatible with Confrey and Smith's (1995) description have characterized quadratic growth as a student envisioning changing rates of change and identifying that the rate of change of the rate of change is constant. For example, Ellis (2011a, 2011b) showed that middle school students can identify constant second differences to realize the quadratic growth in the area of a growing rectangle.

With respect to exponential growth, researchers have used two of the aforementioned characterizations of covariational reasoning to characterize productive meanings of exponential growth. Confrey and Smith (1994, 1995) leveraged their operationalization of covariational reasoning to describe exponential growth as a juxtaposition of values of one variable changing in arithmetic progression with values of a second variable changing in geometric progression. They reported on students interpreting a table of values by calculating the ratio of successive values of one variable for constant unit changes in the other variable to conceive of exponential growth. Confrey and Smith proposed this conceptualization of a constant multiplicative rate as a foundational idea to approach exponential growth.

In contrast to comparing successive values, Thompson (2008) emphasized, "a defining characteristic of exponential functions is that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument" (p. 39). Drawing on this view of exponential growth, Castillo-Garsow (2012) presented tasks in the context of interest bearing bank accounts to high school students and reported on one student who, consistent with Thompson's description, conceived that the rate of change of the value of the account at a moment was proportional to the value of the account at that moment.

Drawing on both Confrey and Smith's (1994, 1995) and Saldanha and Thompson's (1998) characterizations of covariation, rate, and exponential growth, Ellis and colleagues (2015) examined the activity of three eighth grade students who developed understandings of exponential growth by reasoning about the height of a plant changing over time. The students reasoned covariationally to conceive exponential growth as the coordination of multiplicative growth of height values for constant unit changes of time, through numerical tabular arrangements. Eventually students were able to make these comparisons for non-constant changes in time.

Although the aforementioned researchers noted that students at various ages are capable of reasoning covariationally to develop understandings of exponential growth, such understandings may not arise naturally from school experiences. For instance, Strom (2006) engaged in-service secondary mathematics teachers in a series of tasks she conceived to be related to exponential growth. She noted that a majority of teachers in her study had difficulties coordinating the images of two quantities changing together and concluded that covariational reasoning was minimal in most teachers' responses. Strom's study highlights that although students can develop understandings about exponential and quadratic growth by middle school, experienced teachers do not necessarily have this reasoning immediately available to them.

We note there is dearth of literature examining teachers' (or students') conceptions of quadratic growth and exponential growth in relation to reasoning covariationally. Moreover, there are no investigations we are aware of examining how teachers' may differentiate between the two growth patterns by reasoning covariationally. For instance, even if a teacher can engage in MA3 as described by Carlson et al. (2002) to determine that Quantity A increases at an increasing rate with respect to Quantity B, how might that teacher determine if Quantity A grows quadratically, exponentially, or in some other pattern with respect to Quantity B? Therefore, in addition to adding to the literature on teachers' covariational reasoning and understanding of quadratic and exponential growth, our study aims to better understand how students and teachers can develop more sophisticated understandings of these growth patterns.

Pilot Study Methods and Task Design

The first author conducted four individual task based semi-structured clinical interviews (Clement, 2000) that lasted for 60-90 minutes with in-service high school mathematics teachers. The teachers volunteered to participate from a convenience sample accessible to the researchers. Each teacher had a minimum of ten years teaching experience and had taught a variety of high school math courses.

Carlson et al.'s (2002) framework informed the design of the *Two Quadrilaterals* task which is an adaption of tasks implemented in previous studies that investigated students' reasoning about rate (Johnson, 2012) and quadratic growth (Ellis, 2011b). In this applet, we provided two sliders. Whereas the longer, pink, slider allowed teachers to animate the two quadrilaterals (one in blue and the other in brown) the shorter, red, slider allowed teachers to change the increment the longer slider changed by, thus allowing both seemingly continuous and discrete growth of the two quadrilaterals (see Figure 1 for several screen shots of the task). At the start, the quadrilaterals are congruent. As the pink slider drags to the right, each side length of the blue quadrilateral increases proportionally with respect to the slider's position and thus the area of the blue square can be represented by quadratic growth. As the pink slider drags to the right, the brown quadrilateral doubles in size for each unit change in the slider by first having its width double then its height double, and so on and thus the area grows exponentially. We intended to examine how teachers' may conceptualize and compare the growths of each quadrilateral. The interviewer prompted the teachers to consider how the *areas* of each quadrilateral covaried with the *pink slider*.

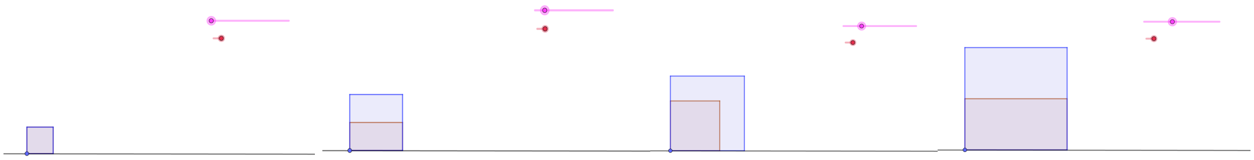


Figure 1. The first four jumps of the Two Quadrilaterals task.

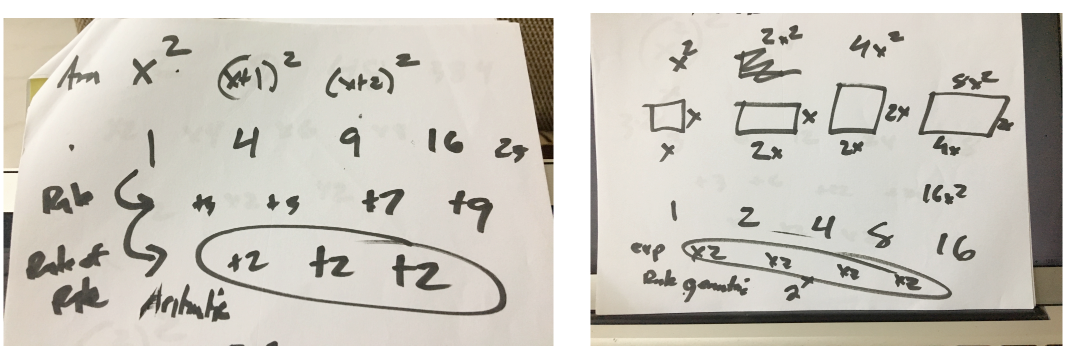
Results

We first present an example of a teacher who explicitly described exponential and quadratic growth when addressing the *Two Quadrilaterals* task. We then briefly synthesize the other teachers' responses to highlight other ways of reasoning the task elicited.

Rick's Ways of Reasoning: Considering Changes to Determine a Relationship

Rick was the only teacher to make statements regarding the type of growth exhibited by the areas of the two quadrilaterals as the slider (or side length) increased. He coordinated how the areas and the slider (or side length) covaried in terms of direction of change (MA2) and the amounts of change (MA3). He then introduced numerical values in order to further analyze and differentiate between the growths of the areas of the two quadrilaterals.

For the blue quadrilateral, Rick noted that both the side lengths increased by one unit if he moved the slider by one unit and claimed "the area is increasing by whatever that side is squared." He considered the initial side length to be 'x' units and described that the areas of the growing square would be x^2, $(x+1)^2$, $(x+2)^2$. Rick next assumed x to take the value of 1, calculated the areas to be 1, 4, 9, 16 and 25, found the differences between these numbers, and also their second differences. Circling the second differences (see Figure 2a), Rick explained "this is the rate of the rate. So the rate of the rate is constant. It tells me it is quadratic." We infer from Rick's explanation that he understood quadratic growth in ways compatible with the characterizations of Ellis (2011a) and Lobato et al. (2012); Rick understood quadratic growth is defined by a relationship such that the rate of change of the rate of change is constant.



(a) (b)

Figure 2. Rick's work describing the growth of area of (a) the blue quadrilateral and (b)the brown quadrilateral.

For the brown quadrilateral, Rick noted that one side length alternately doubles as he moved the slider by one unit. Similar to his aforementioned work (see Figure 2b), Rick described that the area of the brown quadrilateral would be x^2, $2x^2$, $4x^2$, $8x^2$ and $16x^2$ at the first five unit values of the slider, assumed x to take the value 1, and calculated the areas as 1, 2, 4, 8 and 16. He identified that for a unit change in the slider, each consecutive area is double the previous area and stated, "the first rate is constant multiplication which tells us it is an exponential growth. The rate is constant." We infer from Rick's activity that he was coordinating the ratio of the area of the quadrilateral with equal changes in the slider to explain exponential growth. His explanation is compatible with Confrey and Smith's (1994) description of exponential growth as having a constant multiplicative rate.

We note that for each quadrilateral, Rick first engaged in the mental actions described by Carlson et al. (2002) in order to conceive that the area increased at an increasing rate with respect to the incremental changes in the slider. He then introduced numerical values for the side length and the corresponding areas to make the distinction between the two growth patterns.

Other Ways of Reasoning

In contrast to Rick's activity, two other teachers were able to correctly identify relationships without explicitly examining the underlying growth patterns. Consistent with pre-service teachers' activity reported on elsewhere (Stevens et al., 2015) these teachers attempted to either derive a formula or recall facts from memory to define the relationships they conceived in the situation. For example, Aman established that the differences in the areas of the blue quadrilateral can be represented by the rule $(2n\text{-}1)\,x^2$ where n is a natural number, but did not describe any pattern in the second differences of this relationship. Similarly, for the brown quadrilateral, he established that the areas can be represented by the formula $2^{(n-1)}\,x^2$. Although Aman successfully determined rules to describe patterns in the growth of the areas, he did not elaborate on what these growth patterns meant with respect to specific function classes.

As another example, David described that the blue quadrilateral remains a square when the slider is dragged and drew the graph of $y = x^2$. However, David did not justify why the graph would be an appropriate representation of the situation. We hypothesize that David recalled from facts that the area of a square can be represented by $y = x^2$ where x represents side length and y represents area. In both this and Aman's example, we note that, we do not make any claims regarding the teacher's meanings with respect to quadratic or exponential growth; each teacher may have been able to differentiate between exponential and quadratic growth but did not experience any need to discuss these patterns when addressing the task.

Preliminary Implications and Intended Questions

We note, that Rick first engaged in the mental actions described by Carlson et al. (2002) before providing hypothetical numeric values which supported him in identifying the different growth patterns characterized by previous researchers (Confrey & Smith, 1994; Ellis, 2011b; Lobato et al., 2012). This is consistent with the productive interplay between different ways to reason covariationally as described by Paoletti & Moore (2017). Further, despite being able to animate the quadrilaterals (seemingly) continuously, none of the teachers seemed to naturally leverage continuous reasoning as described by Saldanha and Thompson (1998) when addressing this particular task. Finally, a limitation of our task was that several teachers successfully produced known rules which limited our ability to make inferences regarding their understanding of the underlying growth patterns represented by the two quadrilaterals. This raises several questions for our follow-up study.

Should we expect pre-service teachers to respond differently to similar tasks than in-service teachers? Why or why not? What other task situations would lend themselves to supporting teachers in distinguishing between quadratic and exponential growth? What other constructs (e.g., smooth/chunky reasoning) may be useful to consider when designing tasks and analyzing data?

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