Points of Connection to Secondary Teaching in Undergraduate Mathematics Courses

Nicholas H. Wasserman
Teachers College, Columbia University

Rina Zazkis
Simon Fraser University

Erin Baldinger
University of Minnesota

Ofer Marmur
Simon Fraser University

Eileen Murray
Montclair State University

Prospective secondary mathematics teachers frequently take as many (or more) mathematics courses from a mathematics department as they do methods courses from an education department. Sadly, however, prospective secondary teachers frequently view their mathematical experiences in such courses as unrelated to their future teaching (e.g., Zazkis & Leikin, 2010). Yet there is some optimism that having instructors alter their instructional approaches in such mathematics courses can enhance such experiences to be a positive part of their preparation for teaching. This theoretical report elaborates on four points of connection to secondary teaching that can be made in undergraduate mathematics courses, illustrated via examples from abstract algebra, and organized along a spectrum of intended implications on secondary teaching. The purpose is to provide a theoretical bridging between instructional approaches in undergraduate mathematics and aspects of secondary teaching practice.

Keywords: Secondary teacher education, Content knowledge, Teacher's mathematical education

Over the past century, mathematicians and mathematics educators have weighed in about the preparation of secondary teachers. On the one hand, secondary mathematics teachers need a sufficiently deep and robust knowledge of mathematics to teach secondary content; on the other hand, (strictly) mathematical ideas are not the only aspect for which secondary teachers need preparation. Teaching is a notoriously complex profession; teacher education, then, is that much more complex.

In this theoretical paper, we explore some of these issues as they relate particularly to instruction in undergraduate mathematics courses. We do so because secondary mathematics teachers are frequently required to be mathematics majors; the main point being that a significant portion of their teacher preparation program consists of content courses in a mathematics department. These include courses such as abstract algebra – which is the primary course we draw on in our examples in this report. This paper aims to explore the following questions: What are ways in which instruction in undergraduate mathematics courses such as abstract algebra, historically, have made connections to secondary teaching? What are other ways in which instruction in undergraduate mathematics courses such as abstract algebra might make connections to secondary teaching? We consider the first question by synthesizing extant literature; we explore the second through the use of intentionally selected examples from current teacher education efforts.

Two Connections to Secondary Teaching in Mathematics Courses in Extant Literature

In exploring and synthesizing extant literature, we attempt to make clear from the outset one of our assumptions. Namely, we aimed to identify common ways in which undergraduate mathematics course instructors have attempted – explicitly or implicitly – to make their content relevant to secondary teacher preparation. As Wasserman (2018) described: to make their *nonlocal* content relevant not only to the *local secondary mathematics* but (in some way) to the *teaching* of *local secondary mathematics*. That is, what we report on below could be conceived of as potential actions an undergraduate mathematics instructor might take that could serve as a point of connection to teaching secondary mathematics. Essentially, the two points of connection described below – content connections and modeled instruction connections – stem from broad syntheses of literature from secondary teacher education and from research in undergraduate mathematics education studies.

Content Connections

One of the most influential, and innovative, scholars to consider content courses for secondary teachers was Felix Klein. Amongst other things, Klein (1932) pointed out what he described as a "double discontinuity" for secondary teachers. The first discontinuity was that the study of university mathematics did not develop from or suggest the school mathematics that students (i.e., future teachers) knew. That is, the teaching of, say abstract algebra, did not draw on or remotely resemble the algebra they had learned previously, which made learning it more difficult. Klein's second discontinuity was a disconnect for these future teachers in returning back to school mathematics, where the university mathematics appeared unrelated to the tasks of teaching school mathematics. That is, the abstract algebra they learned did not seem useful for teaching algebra to secondary students. Despite the fact that his observation goes back about 100 years, it still rings true today. Undergraduate students, including prospective teachers, often find their experiences in university mathematics courses difficult (e.g., Dubinksy, Dautermann, Leron, & Zazkis, 1994), and secondary teachers find them disconnected from their future classroom teaching (e.g., Zazkis & Leikin, 2010). Klein's primary resolution to this dilemma was to make explicit the mathematical connections that existed between school and university mathematics – an approach he coined as "elementary mathematics from an advanced perspective." Klein's approach – content connections as a point of connections to teaching – is still important today.

The *Mathematical Education of Teachers (I and II)*, more recent reports published by the Conference Board of the Mathematical Sciences (CBMS, 2001; 2012) that outline recommendations for mathematical content and courses to be included in teacher education programs, adopts a similar stance to Klein. They suggest, for example, that "[i]t would be quite useful for prospective teachers to see how $\mathbb{C}$ can be "built" as a quotient of $\mathbb{R}[x]$ and, more generally, how splitting fields for polynomials can be gotten in this way" (CBMS, 2012, p. 59). Mathematicians and secondary teacher educators agree that these mathematical connections are important; textbooks about mathematics for high school teachers (Bremigan, Bremigan, & Lorch, 2011; Sultan & Artzt, 2011; Usiskin et al., 2003) frequently explore such connections between the content of undergraduate mathematics and how it relates to the mathematics studied in secondary school.

The general premise is that studying undergraduate mathematics serves to deepen, and more rigorously confirm, the specific mathematical ideas secondary teachers will teach. In terms of teaching, though, the intended implication is that secondary mathematics teachers will have a normatively correct understanding of secondary mathematics topics and be able to convey these concepts accurately to their students. Such development is particularly important in mathematics writ large, given that mathematical ideas explored earlier in school are often re-explored later with increasing mathematical sophistication. That is, mathematical ideas build on themselves. Secondary teachers need to do a sufficiently good job teaching school mathematics to secondary students since, in undergraduate mathematics, these ideas will continue to be developed.

Coherent concept development and points of mathematical connection, at least ostensibly, serve a specific purpose in teacher education – a point of connection to secondary teaching.

Modeled Instruction Connections

Perhaps less explicit in the literature, but no less powerful, is a point of connection that might be described as modeled instruction. Undergraduate mathematics instructors have the opportunity to take advantage of the age-old adage, "we teach how we were taught," by teaching in ways they would want their students to teach secondary mathematics. This is likely (and perhaps rightly) not at the fore of an undergraduate mathematics instructor's mind when teaching; but it nonetheless provides another point of connection to teaching. Especially given the observation in teacher education (e.g., Brown & Borko, 1992) that "methods" courses are often insufficient to shift a prospective teacher's future practice to more reform-oriented instruction; many revert to teaching in ways they themselves were taught. In the literature, we see much of this notion of modeled instruction of a point of connection to teaching, implicitly or explicitly, as part of the work of the RUME community. In this literature base, scholars have studied and redesigned undergraduate courses to be more in accord with how students learn and develop mathematical ideas, which aligns with more inquiry- and reform-oriented mathematical instruction.

Frequently steered by the notion of guided reinvention from the instructional design theory of Realistic Mathematics Education (RME) (e.g., Gravemeijer & Doorman, 1999), the RUME community has provided many examples of, and resources for, instruction in undergraduate mathematics courses that align with reform-oriented instruction. (In abstract algebra, see Larsen, et al. (2013); in linear algebra, see Wawro, et al. (2013); in calculus, see Oehrtman, et al. (2014); etc.) Often, by building on student thinking, these instructional approaches help alleviate aspects of the first discontinuity Klein observed. But also, as argued by Cook (in press), such instructional approaches, which build on student thinking, provide a model of good pedagogical practices for secondary teachers. This portion represents another connection to secondary teaching via modeled instruction. By instructing in particular ways, students learn mathematics in new ways, which potentially shapes the way they believe that mathematics instruction should occur.

Identifying Two Other Connections to Secondary Teaching in Mathematics Courses

Essentially, the literature has pointed out two sides in what might be regarded as a spectrum of connections (Figure 1a). On one side are connections that are "mathematical" in nature – content connections which primarily aim to influence the mathematical aspects of one's instruction. On the other side are those that are "pedagogical" in nature – modeled instruction connections which primarily aim to influence the pedagogical aspects of one's instruction. Indeed, mathematics and pedagogy are two important, perhaps obvious, lenses through which to view mathematics teaching. The purpose in placing these two on different sides of one spectrum is not to claim they are disjoint, or even easily separable; rather, it is to highlight that content connections and modeled instruction connections – two "means" by which an undergraduate mathematics instructor might make a point of connection to teaching – have *different* "ends" when it comes to teaching, and also to situate the two other connections discussed in this paper as being *between* these two sides of the spectrum – that is, as having intended "ends" that aim to have influence partly on mathematics and partly on pedagogy. Indeed, part of the premise of this paper is that elaborating on different points of connection to teaching that could be made in undergraduate mathematics instruction is good because having an arsenal of "means" (not just two – or three or four, for that matter, but many) and a variety of "ends" both expands and gives substance to the complexity of mathematics teaching.

Figure 1a. Connections to secondary teaching in mathematics courses in extant literature

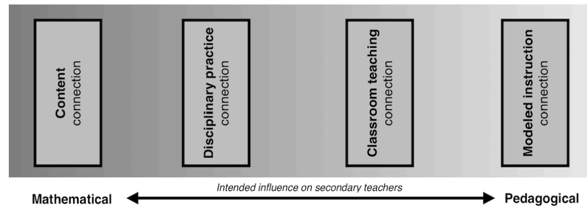

Figure 1b. Four connections to secondary teaching in mathematics courses along a spectrum of implication

In recent work, Wasserman (in press) elaborated on two other kinds of connections to secondary teaching that might exist – ones that fill in areas on the spectrum above, serving part mathematical and part pedagogical ways of connecting to secondary teaching (Figure 1b). Now, the purpose in elaborating on these other two kinds of connections is not to speculate some as better than others, but rather to add to the list of different points of connections to secondary teaching and to organize them along a spectrum of intended influence to be more explicit about their role in connection to teaching. In what follows, we elaborate on these two other kinds of connection to secondary teaching, using examples from abstract algebra: i) disciplinary practice connections; and ii) classroom teaching connections.

Disciplinary Practice Connections

By a *disciplinary practice connection* being the point of connection to secondary teaching, Wasserman (in press) meant that the same kind of disciplinary practice that one engages in while studying undergraduate mathematics can also be engaged in while studying secondary mathematics. Such practices might include defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, et al., 2005), or what Cuoco et al. (1996) termed mathematical habits of mind. Indeed, the processes that one engages in while "doing" undergraduate mathematics are related to some of the important mathematical practices that have been identified and stated as explicit learning goals for school mathematics – e.g., NCTM's (2000) process standards, or CCSSM's (2010) mathematical practice standards.

Hence, these kinds of connections serve a dual purpose. First, they serve a mathematical purpose. By becoming better "doers" of mathematics, secondary teachers have a better grasp on the discipline itself – i.e., the epistemological nature of mathematics, etc. Second, though, these connections also serve a pedagogical purpose. That is, by learning more about what doing mathematics means, there is a hope that secondary teacher's pedagogical choices will, in fact, engage their own students in these forms of thinking and doing. Thus, while these may be primarily about an improved mathematical sensibility (more on the mathematical end of the spectrum) there is also an embedded pedagogical implication (at least partially toward the pedagogical end of the spectrum). Indeed, one of the three perspectives of the Mathematical Understanding for Secondary Teaching (MUST) framework (Heid & Wilson, 2015) is mathematical activity; that how one is engaged in doing mathematics can be a point of connection to the practice of teaching mathematics. The MUST framework highlights mathematical noticing, reasoning, and creating as activities whereby one's experience in undergraduate mathematics courses can parallel the work of teaching school mathematics (Zbiek & Heid, in press).

An example of disciplinary practice connections from an abstract algebra course. In a recent study, Baldinger (in press) used a multiple case study approach to describe four pre-service secondary teachers' learning of mathematical practices from an abstract algebra course.

The abstract algebra course was designed specifically for an audience of secondary teachers, and, although there were certainly content connections (e.g., fundamental theorem of algebra) and modeled practice connections (e.g., problem solving), one of the primary instructional approaches in the course revolved around disciplinary practice connections. That is, the instructor was explicit in describing disciplinary practices, such as, "That's one teaching tactic I have for a challenging proof. I try to come up with a simple example where all the reasoning for the general case is right there. A generic example... A generic example illustrates a line of reasoning that generalizes." Indeed, students were provided intentional opportunities to practice using such generic examples as they solved problems during the course.

Using a pre-post analysis from task-based interviews, Baldinger (in press) found that the pre-service secondary teachers had become more expert in engaging in mathematical practices. That is, when given a novel mathematical problem, the mathematical activities and lines of reasoning they engaged in better reflected such disciplinary practices after having taken the abstract algebra course. Furthermore, the specific disciplinary practices they engaged in reflected those that the instructor had made very explicit during the course. Although one would hope that taking undergraduate mathematics courses would improve students' mathematical activities, students often emerge unable to engage in core practices such as proving (e.g., Weber, 2001). In this study, being explicit about disciplinary practices, with opportunities to practice using them in class, seemed to help the pre-service teachers incorporate such practices into their own mathematical activity. Additionally, three of the four participants also reported that they saw specific connections between the course and their own (future) teaching. The connections they described primarily suggested that they intended to incorporate such disciplinary practices into their own instruction.

Classroom Teaching Connections

In terms of a *classroom teaching connection* being the point of connection to secondary teaching, Wasserman (in press) meant that some connection regarding the content of undergraduate mathematics was being applied to a specific secondary teaching situation. That is, the undergraduate mathematics served as a means to motivate particular and specific kinds of pedagogical actions in the classroom. For example, Wasserman and Weber (2017) explored how the study of proofs of the algebraic limit theorems can be applied to situations when secondary teachers interact with secondary students about rounding and operating on rounded values.

The primary implication in these kinds of connections is about shaping a teacher's pedagogical response to a specific teaching situation – which may be about designing problems with particular characteristics, about responding to students, about sequencing activities, etc. However, such situations are also mathematical, in the sense that the intended point to exploring the teaching situation also includes applying and incorporating mathematical (and not strictly pedagogical) ideas. That is, one's pedagogical response to a situation is explicitly informed by some mathematical idea or mathematical analysis.

An example of classroom teaching connections related to abstract algebra content. In a recent paper, Zazkis and Marmur (in press) elaborated on several instructional situations in secondary mathematics where teachers' knowledge of group theory could serve to shape teaching – namely, their responses to situations of contingency.

School mathematics requires that students understand different sets of numbers (i.e., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) as well as basic operations on those numbers, (i.e., $+, - \times, \div$). In particular, one goal of school mathematics is to help students understand that, as the sets of numbers "expand," the ways in which we conceptualize the operations might also need to expand. That is, while multiplication on the natural numbers can be viewed as "X groups of Y," this idea makes less sense with rational, real, and complex numbers. So although students might "know" multiplication, their notion of multiplication must also adapt somewhat to take into account the kinds of numbers under consideration. Responding to student questions about "What does $\frac{1}{2} \times \frac{3}{5}$ mean?" or "What does $(2 + 3i) \div (5 + i)$ mean?" takes paying attention to, and pointing out, the differences in meaning of an operation depending on the numbers involved.

Here, an experience with programming may be a useful source for understanding the pertinence of group theory. In MAPLE, the command *isprime* tests for whether the input is prime. Yet, in an earlier version of MAPLE, the command *isprime*(14/2) returned "false" (i.e., not prime) – a strange conclusion indeed. It turns out, *isprime* was defined for integer inputs and division was defined for rational inputs. Individually, both of these are sensible: all primes belong to the integers; division makes the most sense with rational numbers because it then maintains the property of closure – dividing rational numbers yields rational numbers. Yet, in combination, MAPLE took 14/2 to mean the rational number 7.0, and not the integer 7 – and it reported the rational number 7.0 to be not prime since it was not an integer. (This bug has since been corrected in MAPLE.) Experiencing this sort of dissonance from a programming environment, and as connected to ideas in abstract algebra, can help teachers develop the ability to attend to ideas of mathematical importance in situations of contingency – e.g., recognizing the importance of different number sets in conceptualizing multiplication with rational numbers or pointing out the importance of closure in defining complex division.

Discussion

The aim of this theoretical report is to provide some initial organizational framing to different points of connection to secondary teaching – especially ones in which undergraduate mathematics instructors might incorporate into their own instruction. We see this as contributing in two aspects. First, although extant literature in the field has explicitly emphasized content connections and, more implicitly, underscored modeled instruction connections, we have identified and exemplified two others: disciplinary practice connections and classroom teaching connections. Second, organizing these four points of connection along a spectrum helps indicate what kind of influence these might have with respect to prospective teachers' instruction. In particular, they provide an ability to be more explicit about how attempted connections made in undergraduate mathematics course might relate to teaching.

We also offer some insights based on the specific examples used in this report. First, from the disciplinary practice connections example, we see that an instructor's choice to be explicit about disciplinary practices during their instruction, and to give students the opportunity to engage in those disciplinary practices during class, appears to have been critical to helping the teachers in the study become more expert in incorporating such practices into their own mathematical thinking and problem solving. We regard being explicit as an important consideration for disciplinary practice connections: without such naming of particular activities, students may miss the generality of a disciplinary practice and the ways in which it gets enacted across a multitude of settings. Second, from the classroom teaching connections example, we see that problems which intentionally mix things may be particularly productive for learning. The cognitive conflict that stemmed from *isprime*(14/2) being "false" required interrogating issues of definition and of closure; not only might we use similar strategies in helping secondary teachers develop additional mathematical awareness in situations of teaching, but we might also discuss pedagogical strategies that leverage cognitive conflict in similar ways to help students themselves attend to (and appreciate) such mathematical nuances and complexities.

Lastly, we discuss some limitations and further reflections. In particular, our theoretical framing has paid particular attention to "mathematical" and "pedagogical" aspects of secondary instruction. This is some ways is a natural starting point – mathematics and pedagogy are intrinsically important. However, there are certainly other important areas of instruction that merit consideration as well – including affective implications, belief systems, issues of equity, etc. How instruction in undergraduate mathematics courses can intentionally make points of connection to other aspects of instruction is an interesting question, worthy of further consideration. In addition, it may be that the four points of connection described in this report also inherently attend to some of these other areas of instruction as well. Regardless, identifying and leveraging theory that merges instructional choices that can be made in the teaching of undergraduate mathematics, with the kinds of implications for secondary teaching that are related to such choices, is an important step in helping to make secondary teachers' experiences in undergraduate mathematics a more meaningful component of their teacher preparation and development process.

References

Baldinger, E. (in press). Learning mathematical practices to connect abstract algebra to high school algebra. In N. Wasserman (Ed.), *Connecting abstract algebra* to *secondary mathematics,* for *secondary mathematics teachers* (pp. XXX), Research in Mathematics Education Series. Springer.

Bremigan, E. G., Bremigan, R. J., & Lorch, J. D. (2011). *Mathematics for secondary school teachers*. Washington, D.C.: Mathematical Association of America (MAA).

Brown, C., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 209-239). New York: Macmillan.

Common Core State Standards in Mathematics (CCSSM). (2010). Retrieved from: http://www.corestandards.org/the-standards/mathematics

Conference Board of Mathematical Sciences (CBMS). (2001). *The mathematical education of teachers.* Retrieved from: https://www.cbmsweb.org/archive/MET_Document/index.htm

Conference Board of Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II.* Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Cook, J. P. (in press). Monster-barring as a catalyst for bridging secondary algebra to abstract algebra. In N. Wasserman (Ed.), *Connecting abstract algebra* to *secondary mathematics,* for *secondary mathematics teachers* (pp. XXX), Research in Mathematics Education Series. Springer.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15*(4)*,* 375-402.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267-305.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*, 111-129.

Heid, M. K., Wilson, P. S., with Blume, G. (Eds.) (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations.* Charlotte, NC: IAP.
Klein, F. (1932). *Elementary mathematics from an advanced standpoint: Arithmetic, Algebra, Analysis* (trans. Hedrick, E. R. & Noble, C. A.). Mineola, NY: Macmillan.
Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *Journal of Mathematical Behavior, 32*(4), 693-711.
National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: Author.
Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *Journal of Mathematical Behavior, 33*(1), 131-148.
Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A view of advanced mathematical thinking. *Mathematical Thinking and Learning, 7*(1), 51-73.
Sultan, A., & Artzt, A. F. (2011). *The mathematics that every secondary school math teacher needs to know.* New York, NY: Routledge.
Usiskin, Z., Peressini, A., Marchisotto, E. A., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective.* Upper Saddle River, NJ: Pearson.
Wasserman, N. (in press). Exploring advanced mathematics courses and content for secondary mathematics teachers. In N. Wasserman (Ed.), *Connecting abstract algebra* to *secondary mathematics,* for *secondary mathematics teachers* (pp. XXX), Research in Mathematics Education Series. Springer.
Wasserman, N. (2018). Knowledge of nonlocal mathematics for teaching. *Journal of Mathematical Behavior, 49*(1), 116-128.
Wasserman, N., & Weber, K. (2017). Pedagogical applications from real analysis for secondary mathematics teachers. *For the Learning of Mathematics, 37*(3), 14-18.
Wawro, M., Rasmussen, C., Zandieh, M., & Larson, C. (2013). Design research within undergraduate mathematics education: An example from introductory linear algebra. In T. Plomp & N. Nieveen (Eds.), *Educational design research – Part B: Illustrative cases* (pp. 905-925). Enmschede, the Netherlands: SLO.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.
Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*(4), 263-281.
Zazkis, R., & Marmur, O. (in press). Groups to the rescue: Responding to situations of contingency. In N. Wasserman (Ed.), *Connecting abstract algebra* to *secondary mathematics,* for *secondary mathematics teachers* (pp. XXX), Research in Mathematics Education Series. Springer.
Zbiek, R. M., & Heid, M. K. (in press). Making connections from the secondary classroom to the abstract algebra course: A mathematical activity approach. In N. Wasserman (Ed.), *Connecting abstract algebra* to *secondary mathematics,* for *secondary mathematics teachers* (pp. XXX), Research in Mathematics Education Series. Springer.