Intuition and Mathematical Thinking in a Mathematically Experienced Adult on the Autism Spectrum

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In this report, I examine the use of intuition by a mathematically experienced adult on the autism spectrum given a paradoxical mathematical problem involving infinity. I compare both his level of use of intuition and the importance he places on it against results from students in the general population. Interview results combined with previous data suggest that students on the autism spectrum are less likely to use approaches based in intuition, place less importance on intuitive ideas compared to other explanations, and may also have different views of the nature of intuition. Analysis of possible reasons for showing these differences and implications for teaching and further autism-related research are presented.

Keywords: intuition, mathematical paradoxes, autism

My research attends to mathematical problem solving by adults on the autism spectrum (with a formal diagnosis), particularly those with a relatively strong background in mathematics. In this report, I focus particularly on the case of one student's work on the Ping-Pong Ball Conundrum, a problem of infinity (Mamolo and Zazkis, 2008). I use this problem to highlight characteristics of intuition used in problem solving and how the use of intuition can differ for people on the autism spectrum in both nature and frequency. This can help to both examine the use of intuition in mathematics generally and to examine characteristics related to autism.

Brief Overview of Autism-Related Research in Mathematics Education

There is a wide range of conceptions of what being on the autism spectrum means, including various academic and clinical definitions. The Autistic Self Advocacy Network (2014), the leading autism advocacy group run by people who are themselves autistic (and identify as such) states that autism is a neurological difference with certain characteristics, each of which is not necessarily present in any given individual on the autism spectrum. These include differences in sensory sensitivity and experience, atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. They also list "different ways of learning" and particular focused interests (often referred to as 'special interests'), which are especially relevant for research in education. Of those characteristics, it is primarily the existence of special interests and the differences in language use and social interaction that are used as diagnostic criteria by the fifth edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-5).

Much of the research currently done on mathematics learning in people on the autism spectrum is focused on young children (e.g., Iuculano et al., 2014; Klin, Danovitch, Mers & Volkmar, 2010; Simpson, Gaus, Biggs & Williams, 2010) or looks at mostly arithmetic. There is also a notable strain of work done on the population of research mathematicians (e.g., Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007; James, 2003), but very little attention is paid to groups in the middle (mainly high school and college students, or adults other than career mathematicians). This is a gap which I have sought to help fill with my own research, including the particular selection which I present here.

Theoretical Framework

My theoretical framework is based partially in Vygotskian theory, particularly Vygotsky's (1929/1993) conception of overcompensation. Vygotsky explained this initially in a framework of physical overcompensation, such as a kidney or lung necessarily strengthening when the other one is missing or by analogy to vaccination. He argued that overcompensation also occurred in psychological development, both in its general course and in particular in the presence of various disabilities. While my views are informed by the Vygotskian framework, there are some issues with using it directly. Some parts that are particularly relevant in autistic people, such as the ideas about atypical development and concept formation, particularly concern things that have already must have occurred far before starting university coursework, and thus cannot be observed in my interview subjects. The examination of inner speech also has difficulties; Vygotsky himself used children whose inner speech had not yet fully developed in his clinical experimentation on the subject. Thus, while those ideas from Vygotsky inform my views, additional constructs were required for the data analysis; in this case, the main one is thinking regarding intuition.

In many contexts, the erroneous conclusions produced by students and the resistance to the mathematically valid solution are identified with forms of intuition. In Fischbein's (1979) use of the idea, intuition is separated into different categories, particularly "primary intuition" (developed outside of a systematic instructional setting) as opposed to "secondary intuition" (developed in a systematic instructional setting). The division of categories here has similarities to Vygotsky's distinction between everyday and scientific concepts, and I find it reasonable to consider the primary and secondary intuition used by Fischbein as identifying intuitive reasoning related to everyday or scientific concepts, respectively. Further exploration of intuition by Fischbein (1982) uses a similar division between "affirmatory intuitions" and "anticipatory intuitions", focusing primarily on the former. In this division, affirmatory intuitions are those that are "self-evident [and] intrinsically meaningful", which again stands outside the systematic instructional context.

In the context of other works, it is the primary and affirmatory definitions that are closest to what is typically meant when 'intuition' is named but not explicitly defined, which is useful for situating other work which mentions intuition but does not focus on it. Fischbein also argues for the importance of using intuitive ideas, which includes but is not limited to correcting those intuitive ideas which would otherwise lead to error. Here, his main focus is on developing intuitive ideas so that they are in accord with the analytic reasoning rather than in conflict. While there are still possible parallels between these theoretical constructs and Vygotskian concepts, the approach suggested by Fischbein, focused on development and adjustment of intuitive ideas, is more constructivist. Since these differences are reflected in neurological differences associated with autism, they would lead to contrasting predictions for the mathematical reasoning of people on the autism spectrum.

Ping-Pong Ball Problem

While I conducted interviews using a variety of problems, in the excerpt here I focus on results from a single problem described in Figure 1.

The Ping-Pong Ball Conundrum
Consider an infinite set of ping-pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are inserted and the '2' ball is removed, and so on. How many ping-pong balls remain in the barrel at the end of the minute?

Figure 1. Statement of the Ping-Pong problem used in the interview.

The accepted mathematical solution here is that there are no balls in the barrel, because for every possible ball, we can find a time after which it has been removed (this is because of the order the balls are removed in, and different orders can lead to different outcomes).

This problem was used by Mamolo and Zazkis (2008) with two groups of students, one undergraduate and one graduate. Both were in courses about fundamentals of mathematics at different levels which involved infinity. They were introduced to this problem after having seen the Hilbert Hotel problem, a simpler problem also involving infinity. In each case, after students' first responses to the problem, they were given the standard solution. Both groups initially gave responses that rejected things in the problem setup that seemed impossible, relating them to real-world facts such as the finite population of Earth. The undergraduate students, who were a more general population of liberal arts and social science students, continued to show resistance to the given mathematical solution in the Hilbert Hotel problem, while the graduate students (who were in a mathematics education program) did not. However, both groups continued to show disbelief in the mathematical solution for the Ping-Pong problem after instruction. One of the more common responses found in both student groups was that there were nine more balls at each step, often giving a 'nine times infinity' response. This highlights the importance of the numerical ordering in the problem, since if the balls were not ordered this way (if they were all simply generic and interchangeable balls, for instance), it would be correct to use the fact that at any step n, there were $9n$ balls in the barrel and to take the limit of that expression as n goes to infinity. In the numbered case, we can view it as essentially an arrangement of processes where a second process 'cleans up after' the first, but calculating the total at each step as $10n - n = 9n$ erases that ordering property. Without a numbering to provide order, that arrangement cannot be made; in that case, there is no information lost with the '$9n$ balls in the barrel' view.

Ely (2011) gave this problem (as the Tennis Ball Problem) to a range of participants from undergraduates who had finished college algebra to mathematics doctoral students and one mathematics professor. He used two versions, comparing the effect of asking "how many balls are left" to asking "which balls are left" (Ely, 2011, p. 8). He found that participants given the 'which' version were more likely to attend to the labeling (ordinal) rather than the total number (cardinal). The 'which' participants also showed more conflict from being presented the accepted mathematical solution of having no balls remaining, although they were still unlikely to accept it. No participants given the 'how many' version accepted the zero-ball solution, while the only participant given the 'which' version to accept it was the mathematics professor.

Disputes about the proper result of this problem have also been shown in publications where researchers discuss their own perceptions of and disagreements about the problem, rather than discussing the perceptions of students. Allis and Koetsier (1991) describe this paradox in terms of super-tasks, defined as "the execution of […] an infinite sequence of acts" (Allis & Koetsier, 1991, p. 189). They argue that this is possible not only in an abstract way, but also in a kinematic one. However, a later discussion by van Bendegem (1994) raises objections to both of these arguments. The objection to the abstract solution is an algebraic argument, while the objection to the kinematic one involves relativistic physical assumptions. The response from Allis and Koetsier (1995) points out that the algebraic argument from van Bendegem does make an assumption of continuity (although van Bendegem asserted that it did not), and raises multiple objections to the kinematic argument. Looking at what is disputed between the authors, it is notable that the two main strands closely parallel the 'nine times infinity' solution and the objection to the real-world possibility of the problem found with the students in the study from Mamolo and Zazkis.

Ultimately, from the range of mathematical experience in responses to this task, we can see that this is a paradox that can incite argument and confusion even at high levels of academic discourse, and that few people at any level are inclined to accept the standard mathematical solution. Also, at multiple levels the nature of disputes and confusion fits into two main categories, one directly related to infinity and continuity and another related to physical properties.

Methodology and Task Details

Given my interest in focusing in-depth on interviews with a small number of people, a method of case studies was a natural fit for my work. Case study focuses on in-depth understanding of the case in question, and only secondarily on generalizations from that understanding. Additionally, while generalization is possible, it is not of the same nature as generalization in other types of research (Stake, 1995). These are sometimes divided between embedded and holistic case studies, where an embedded case study is interpreted as examining a particular feature or subset of the case in question, while a holistic case study does not use such subdivisions (Yin, 2009). In this case, my decision for a holistic case study naturally follows from my neurodiversity-informed view that the nature of being autistic is not a discrete part of the person that can be separated, and thus an embedded design does not apply.

My participant here (who chose the pseudonym Cyrus) was recruited from the community and was in his thirties at the time of interview, holding a bachelor's degree in mathematics and working in computer programming. He received an ASD diagnosis at age 13. I conducted several interviews with him as part of my broader work; the one I focus on here involves the ping-pong ball problem, as described above.

Interview with Cyrus

In the beginning, the interviewer reads the problem to Cyrus and then shows him a printed version of the problem. There are multiple rounds of explanation as the problem is presented to Cyrus before he understands the problem correctly and addresses it. Once he does, this is his response.

C: Okay, but if that keeps going on, then, but we're eventually going to have an infinite number of balls in there, but it just depends on how many, if we go through n successions of this, we're just not going to have the first n in the basket. [*I*: Alright.] So, but it's still an infinite number of balls, we're just not going to have, 1 through n in there.

I: If n goes to infinity, what does not having the first n mean?

C: I don't know. That means we've got no balls in there whatsoever. That sounds like the intuitive answer, but I'm just not certain about that.

I: Hm. Ah, that's the intuitive answer for what reason?

C: Um, that still doesn't sound right, but you'd still, even if n went to infinity, you'd still have an infinite number in there. Yeah, that's what I would say. I don't know what my reasoning is for it, I think just like you could, just like in [the Hilbert] hotel problem you could keep placing more and more in there. Well, okay, something similar to that. In this case you could just keep adding more and more balls in there, even after, going through an infinite number of times. But except, oh my god it doesn't, yet it still seems to kind of contradict itself. Because if n now encompasses all the numbers, and you're taking them all away, then how can you have an infinite number of balls still in the basket? So, I don't know, I'm going to say there's zero balls in there, at that point, if you take the limit as n goes to infinity, I'm going to say there's zero balls. [*I*: Okay.] I know I sort of changed around on that one.

I: Alright. Mm, but at this point would you say that that is what you're sticking with?

C: Yeah, I'm going to stick with that, if n goes to infinity, then you have zero balls left.

I: Okay, and does that seem like, a reasonable sort of thing, or does it seem kind of weird, or, ah, is it something that you would accept?

C: I think it seems weird, but I would accept it.

At first, Cyrus looks at the time intervals, finding the length of the first four and suggesting that they diverge as a series. The interviewer clarifies that the focus of the problem is on the number of balls, and describes what is in the bin during the first two time intervals. Cyrus extrapolates from this that as each time interval progresses, more and more balls will be in the barrel, but the first n will be missing. The interviewer asks what this means as n approaches infinity (which may be considered either the number of steps performed or the number of balls removed: since these are always equal, it is not clear from the statements which conception Cyrus is focusing on), and Cyrus says that means there will be no balls in the barrel. At first he calls this an intuitive answer; thinking further, he switches to there being an infinite number and then back to zero (without any intervention). He describes it as weird, but is willing to accept it.

The brief characterization of the zero answer as intuitive is unusual and may suggest multiple layers of reasoning that Cyrus considers "intuitive". However, while this does not last, the intuitive conclusions are still not held by Cyrus to be particularly important for a final conclusion.

Next, the interviewer presents Cyrus with some alternative arguments made by other students:

I: Okay. And, there are, I think we have them here, ah, couple of arguments that, ah, different students had, ah, trying to work out this particular problem, and I'd like to tell you about a couple of those, and see what you think about them.

C: Okay.

I: Okay, so, one argument about this was that, for each chunk, you're essentially adding nine balls. [*C*: Mm-hm.] So the total amount at any time should be nine times, the amount of chunks you've gone through. [*C*: Mm-hm.] And, but that goes to infinity.

C: Right, okay, so, you add nine but, right, you add nine balls, okay.

I: So does that seem correct or incorrect, and, why? How does that argument sound to you?

C: So, it's like, nine times n, okay. And if n goes to infinity, from that one, the answer would clearly be it just blows up, if n goes to infinity, it would be infinite. You'd have infinite number of balls left in there. So, but it still doesn't seem to make sense to me when I try to actually predict in my head what's going on there, it doesn't really seem to make sense with that. I would say that answer doesn't make as much sense. To me it makes sense as long as you have a finite number of time intervals.

I: Okay. And why doesn't this work in the infinite case?

C: My only reasoning is somehow it doesn't make sense to me, once you've already taken away, essentially, once you've taken away every single natural number, then you can't have anything left.
I: Mm. Okay, well,
C: Yeah, now I'm really confused, I'm just not sure if there's a correct answer to this or not.
I: That's sort of, ah, the second argument, where we ask, okay, if there are balls remaining, ah, all our balls are numbered by natural numbers, [*C*: Mm-hm.] so, if there's some balls remaining, what are they? Name one.
C: Okay. So, if there are some balls remaining, then what are they. Name one. Okay. Um, so then, I don't know, my- my reasoning followed the case where you'd have nothing left at the end.
I: Right. Mm, which- yeah, that's- that's correct. And that's why that works, is it- there aren't any. [*C*: Mm-hm.] You can't name one.
C: Right.
I: This- this is sort of, the, ah, contradiction proof version of proving this. [*C*: Okay.] Where you go, okay, suppose by way of contradiction, that there are some balls left. [*C*: Mm-hm.] Then- since this is a set of natural numbers, it must have a least element. Call it n.
C: Right, okay.
I: But then, in the nth step, we've removed that. [*C*: Mm-hm.] So we don't have that.
C: Right, okay.
I: Contradiction.
C: That's a proof by contradiction, I see.
I: Yeah. Therefore, there are no balls.
C: Mm. Ah, okay. And, okay, but I wouldn't have really thought of that- not in that way, at least, but it makes sense, once you've- go to the next step, you've just removed the one that's remaining.

In this segment, Cyrus is first presented with the argument that, since nine balls are added each time, there should be infinitely many at the end, and asked what he thinks of it. He finds it to make less sense, but after being presented with it, is uncertain if there is a correct answer. He is then presented with the proof by contradiction argument for there being zero balls, and agrees that one cannot name a single ball remaining, though he says he would not have considered it in that way.

While Cyrus' conclusion agrees with the proof by contradiction conclusion, he says that he would not have viewed it that way. This may be related to the first intuitive answer he gave earlier, not having the 'infinity' conclusion as something to start off with as reasonable to contradict, which suggests that Cyrus may view the problem in an unusual way which is more conducive to the ultimately correct solution. In fact, not only does Cyrus not reach that as a conclusion, he unusually characterizes it as making less sense, while most typical students have the opposite view. However, he does agree that such a solution is valid for any finite case.

Analysis

In this particular interview, it appears that Cyrus may not have the unexamined continuity assumption that many students use to extrapolate to the infinite case in this problem as part of his intuition. Alternatively, he may have learned to ignore it. If it is ignored, this is also noteworthy; although Cyrus does have formal mathematical training, others at the same or higher level of formal mathematical training still did not have this response, as found by Ely (2011).

Here and in other interviews, Cyrus tends to have a high level of trust in the truth and consistency of mathematics, not displaying many of the typical objections to paradoxical tasks outlined in prior research. When faced with apparent contradictions, he is more likely to question an intuitive response rather than a formal mathematical result. Additionally, contextual considerations of problems phrased in a 'real-world' physical setting appear to be given less relevance. This also appears to be a logical result of an orientation toward structure, or in Vygotskian terms, systematic over intuitive reasoning.

One overall trend apparent from Cyrus' interviews is his inclination toward algebraic and formal methods of solution, and a distrust of or disinclination toward informal or intuitive methods. This can fit as another form of compensation, where stronger algebraic skills are used in place of weaker geometric or informal ones. However, this combined with interviews with my other participants (Truman, 2017) suggests that any effect of compensation related to autism is more complex and individual rather than people on the autism spectrum all fitting a certain type. Cyrus' forms of compensation here contain points of similarity to other participants, such as the mistrust of intuitive reasoning and inclination toward more systematic justifications. However, there were also differences fitting a broader pattern; in particular, Cyrus skews more heavily toward using algebraic methods and avoiding geometric methods or methods based in physical analogies. For Cyrus' reasoning in this problem specifically, the physical system of balls being removed was never a focus, and physical impossibility was not considered as a notable issue. This is in particular contrast to my first interview participant, who had the opposite pattern (Truman, 2018), although all of my participants showed a strong inclination toward a particular mode of problem-solving.

Conclusions

For many students, it is often difficult for them to trust in systematic over intuitive reasoning when they are in conflict, particularly in the sorts of problems typically called mathematical paradoxes. Thus, this tendency for students on the autism spectrum to rely more on systematic reasoning can be a particular advantage in such situations that most students find difficult, as we see in this particular case. However, instructional approaches that are designed to rely on students' intuitive reasoning or real-world concepts may be less successful for students on the autism spectrum for the same reason, or they may result in unusual responses that would require more instructional attention.

My research findings here are also consistent with the theory that people on the autism spectrum learn in a manner that relies less on prototypes (Klinger & Dawson, 2001) in favor of constructing concepts more systematically. I believe that the examples of problem-solving in the interviews show that this can produce positive results and does not need to be viewed as a deficiency. They also shed more light on cognitive differences of people on the autism spectrum in adulthood, which is particularly important because much of the research done related to autism is done with younger children. They are also consistent with the systemizing theory (e.g., Baron-Cohen, Wheelwright, Burtenshaw, & Hobson, 2007), where systemizing is viewed as an inclination to create or analyze a system based on the formulation of rules. However, this could also come from the mathematical inclinations already known about and sought in the participants. This should not be taken as support for the suggestion by many proponents of the systemizing theory of its opposition with empathizing (viewed here as the recognition of what someone else is feeling), since the nature of the interviews shows very little about any skills in that category, either positively or negatively.

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