Why Don’t Students Check their Solutions to Mathematical Problems?
A Field-based Hypothesis

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This theoretical paper introduces a field-base hypothesis, according to which the intensity and type of an intellectual need that students can experience for checking their solution to a problem might be related to the epistemological status of methods that they employed for solving the problem. The hypothesis emerged from the analysis of a final exam in a first-year course where 421 students worked on four problems in linear algebra. In one of them, 33 students provided evidence of checking their solutions, all of which appeared as educated guesses. No written evidence of checks was indicated in the deductive solutions, in which the students utilized algorithms, procedures, and theorems that were introduced to them in the course. Thus, it might be proposed that problem-solving methods with a low epistemological status (e.g., educated guesses) may instigate the need for checking a solution as a means to compensate for their status.

Keywords: checking solutions, DNR-framework, epistemological status, intellectual needs, problem solving.

Introduction and Literature Review

Let us assume that in her final exam in linear algebra, Rina was assigned with the problem in Figure 1. The solution is far from easy in this case, as it requires fitting together a considerable number of topics that the course covered: systems of linear equations, bases of vectors spaces, column spaces, and that just for the first part! Now, let us assume that Rina has put her course studies to use, which created an opportunity for her to check her own work. Hence come the questions whether she will do the checks, and if yes, how.

Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ 0 & 1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$, and let S be the set of all vectors $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a solution.

(a) Find a basis for S (no need to show that S is a subspace of $\mathbb{R}^3$).
(b) Find the least square solution to $A\bar{x} = \vec{c}$.
(c) Find the corresponding least square error.
(d) Give a non-zero vector that is orthogonal to every vector in S.

Figure 1. An assigned problem.

Research has been approaching such questions through the lens of metacognition and problem solving, when the lion’s share of studies have been conducted in the context of school mathematics (e.g., Cai, 1994; Lucangeli & Cornoldi, 1997; Pugalee, 2004; Schoenfeld, 1992). One line of this research might propose that it would be rather atypical if Rina attempted to check her solutions. For example, in his study with twenty ninth-graders, Pugalee (2004) found that verification – evaluating decisions and checking calculations – was the rarest behaviour compared to the ones that the students exhibited at the orientation, organization, and execution phases in their problem solving. Another line of research might advise Rina to undertake the checks due to the recurrent findings on the relation between verification of solutions and successful problem solving (e.g., Cai, 1994; Lucangeli & Cornoldi, 1997). Malloy and Jones (1998), for instance, found a moderate correlation between problem success of twenty-four students of ages 12-14 and their verification behaviors. The verification in their study was associated with rereading the problem, checking calculations, checking the plan for solution, using another method, and redoing the problem. However, the findings of Mashiach Eizenberg and Zaslavsky (2004) may confuse Rina's decision-making. The participants in this study were fourteen undergraduate students, who initiated a verification of their solutions in nearly two thirds of the cases. Despite students' attempts, however, every second solution remained incorrect.

From the metacognitive point of view, the question of "to check or not to check" a devised solution pertains to how one allocates cognitive and affective resources during problem solving (Schoenfeld, 1992). Verschaffel (1999) maintains that such checks are especially important at the final stages of problem-solving cycles, where solvers need to interpret the outcomes of their work. In Schoenfeld's (1992) terms, checking can be viewed as an instance of monitoring since it is part of one's reflecting on the effectiveness of her problem-solving processes and products. Overall, the acknowledgement of the importance of checking can be traced back to the classical work of Pólya (1945), specifically to the "carrying out the plan" step for solving a problem and "looking back" at the devised solution. When carrying out the plan, Pólya recommends the solver to check and prove the correctness of each move that she undertakes. The "looking back" step, in turn, is instigated by such questions as "can you check the result?", "can you derive the result differently?", and "can you use the result, or the method, for some other problem?" In this way, despite its title, this step is targeted at preparing the solver for the next problem, the solution of which might be easier if she would take the time to critically reflect on the problem that has been solved already.

In this theoretical paper, I present a *field-based hypothesis* on possible relations between contextual affordances that can emerge when one solves a problem and consequent moves that she might undertake for checking her solution. Harel (2017) posits that a field-based hypothesis is

> "suggested by observations of learners' mathematical behaviors in an authentic learning environment, and is explained by cognitive and instructional analyses oriented within a particular theory of learning, but has not, yet, been proved or disapproved by rigorous empirical methodologies in large scale settings" (p. 70).

The DNR-framework is used as a theory of learning in this paper, when its selected constructs are reviewed in the next section. This is followed by a description of an authentic learning environment, in which observations of a large cohort of students were made. An analysis of these observations gives rise to the hypothesis in the last section.

Intellectual Need and Epistemological Justification

In Harel (2008a, b), Guershon Harel introduced a comprehensive conceptual framework "which seeks to understand fundamental problems of mathematics teaching and learning" (Harel, 2013a, p. 3). This epistemologically solid framework has already exhibited its analytical power and usefulness for designing teaching environments (e.g., Harel, 2013a, b, 2017). The framework has been termed with the acronym DNR, which stands for three pillar principles: duality, necessity, and repeated reasoning. The full brunt of DNR goes beyond the scope of this paper, hence, I provide a brief overview of its central constructs that are utilized later on.

The necessity principle grows from the work of Piaget (1985), in which learning is viewed as occurring in situations where one attempts to resolve a mental disequilibrium. Harel

(2017) encapsulates the principle as follows: “For students to learn what we intend to teach them, they must have a need for it, where ‘need’ refers to intellectual need” (p. 75). *Intellectual need* is conceived as a contextualized construct that comes into being in a situation which one experiences as problematic in the sense that her current state of knowledge is insufficient or incompatible and additional piece of knowledge should be acquired in order to reach an equilibrium. Specifically, Harel (2013a) distinguishes between five categories of intellectual needs: *the need for certainty* can emerge when a learner has doubts about the trueness of a particular assertion; *the need for causality* is the need to determine a cause of a phenomenon (i.e. to explain); *the need for computation* pertains to quantifying numeric values that are missing; *the need for communication* is manifested through formulating and formalizing for the sake of conveying and exchanging ideas; finally *the need for structure* is the need to reorganizing one’s knowledge into a logical structure. Kontorovich and Zazkis (2016) offered to enrich this categorization with Koichu’s (2008) principle of intellectual parsimony, which states that when solving a problem, a person can avoid investing more intellectual effort than the needed minimum for obtaining a solution. This principle may be positioned as an intellectual *need for parsimony*, the need which might explain why a particular piece of knowledge has not been constructed.

Harel (2013a) maintains that the notion of intellectual need is tightly connected to *epistemological justification*, which “refers to the learner’s discernment of how and why a particular piece of knowledge came to be. It involves the learner’s perceived cause for the birth of knowledge” (p. 8). In his later work, Harel (2018) offers a typology of epistemological justifications, where one of the types is *apodictic*. This justification pertains to one’s viewing the proving process of the logical implication $\alpha \rightarrow \beta$ either in causality or explanatory terms. An apodictic justification manifests itself when one is interested either in the consequences of α, or in the possible causes of β. Accordingly, α and the whole apodictic chain that leads to β endow a *high epistemological status* in relation to β.

Harel (2013a) emphasizes that intellectual needs are ingrained in all aspects of mathematical practice, which allows the application of his framework to the purposes of this paper. Indeed, α can be associated with an assigned problem, where the solution process constitutes an apodictic epistemological justification that causes and explains the emergence of the final answer β. Thus, the checking of β turns into an act of knowledge construction, through which one might fulfil her intellectual needs.

Observational Environment

The data illustrated comes from written solutions that 421 students submitted as part of their final exam in a first-year mathematics course. The course was delivered at a large New Zealand university and it was intended for undergraduates majoring in computer science, economics, statistics, and finance. For students enrolled in the course, it was their second encounter with university mathematics with a focus on two-variable calculus, differential equations, and topics in linear algebra where the necessary methods for solving Figure 1 were introduced. The course instruction can be described as mostly traditional and lecturer-centred, with some emphasis put on students’ reasoning. For instance, the guidelines for the exam in which Figure 1 was assigned stated, “You must give **full** working and **reasons** for your answers to obtain full marks” (bold in the origin).

The analysis of the solutions that the students submitted consisted of an iterative process with deductive and inductive components (Denzin & Lincoln, 2011), that corresponded with two questions: (i) What types of mistakes do students make in their solutions? (ii) What characterizes the solutions, in which the students provided written evidence of checking their final answers? The analysis started with a review of the correctness of students' submissions, where the ones with mistakes were classified according to the steps that distorted the problem-solving chain. This classification was informed by Movshovitz-Hadar, Zaslavsky and Inbar (1987), who explored common errors that students make in their matriculation exams. After analyzing 860 scripts, the researchers came up with six categories: distorted theorem or definition, technical error, misused data, misinterpreted language, unverified solution, and logically invalid inference. Due to the similarity of the analyses and types of data, these categories were used as a baseline for analyzing students' solutions in this study. At the next stage, a constant comparison technique (Glaser & Strauss, 1967) was employed for characterizing those solutions with written checks. The comparisons were targeted at delineating similarities between the solutions submitted by different students. The emergent similarities were applied for all data corpus to validate that they are characteristic indeed.

Overview of Students' Solutions and Their Checks

Table 1 provides an overview of students' submissions, and it shows that obtaining a final answer cannot be taken for granted in the cohort under scrutiny. Clearly, the written solutions that the students submitted captured only a part of the problem-solving journey that the students undertook. Hence, a lack of a check of a solution provided no evidence of whether and how a student monitored her work (some of the checks could have been carried out mentally, for instance). However, students' submissions of mistaken solutions point at the struggle to check the work, or a missed opportunity to do so. In turn, instances where students provided written checks deserve a special attention.

Table 1. Overview of students' solutions.

	Part (a)	Part (b)	Part (c)	Part (d)
<u>Final Answers Submitted</u>	263 (62.5%)	309 (73.4%)	217 (51.5%)	171 (40.6%)
Correct	131 (49.8%)	145 (46.9%)	151 (69.6%)	40 (23.4%)
Incorrect	132 (50.2%)	164 (53.1%)	66 (30.4%)	131 (76.6%)
<u>Sources of Incorrect Answers</u>				
Mismatch between	44 (33.3%)	5 (3.05%)	13 (19.7%)	56 (42.75%)
a problem and employed method			-	
Methods with distorted steps	63 (47.73%)	6 (3.66%)	20 (30.3%)	13 (9.92%)
Computation mistakes	-	149 (90.85%)	-	19 (14.5%)
No solution process	25 (18.94%)	-	-	19 (14.5%)
<u>Written Checks</u>	-	-	-	33 (19.3%)

Table 1 shows that *all* written checks that the participating students submitted as part of their solutions to Figure 1, appeared as a response to Part (d). These checks encompassed computations of the dot products of the vectors from the basis in Part (a) with the vector which was a candidate for an answer. While every four out of ten computations contained a mistake (see Figure 2 for example), all the checks maintained that the dot product is zero. Accordingly, these checks can be viewed as enactments of an appropriate strategy, in which vectors' orthogonality has been attempted to be verified with a critical attribute that was used in the course for defining the concept.

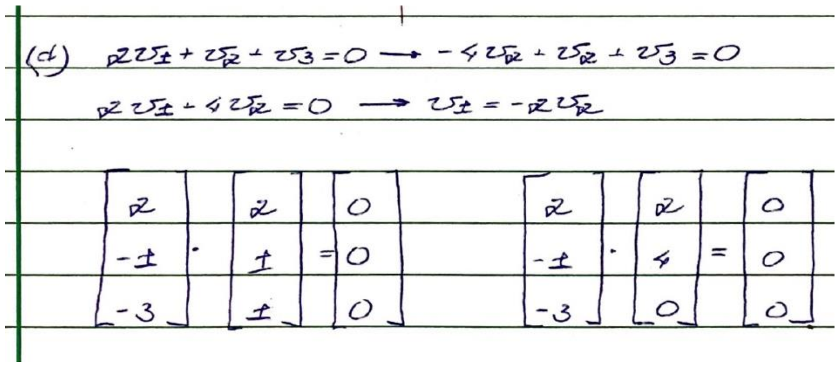

Figure 2. Example of an educated guess in Part (d).

One notable characteristic that was identified among all the solutions with written checks is that the candidates for orthogonal vectors were not devised with structured problem-solving methods that were studied in the course. Figure 2 exemplifies one third of such solutions, where the students started with a system of linear equations that the coordinates of the orthogonal vector were expected to satisfy. Yet, the equations were not solved fully and an orthogonal vector was introduced at some point. In the remaining solutions, the students started by declaring which vector is orthogonal (see Figure 3 for an example).

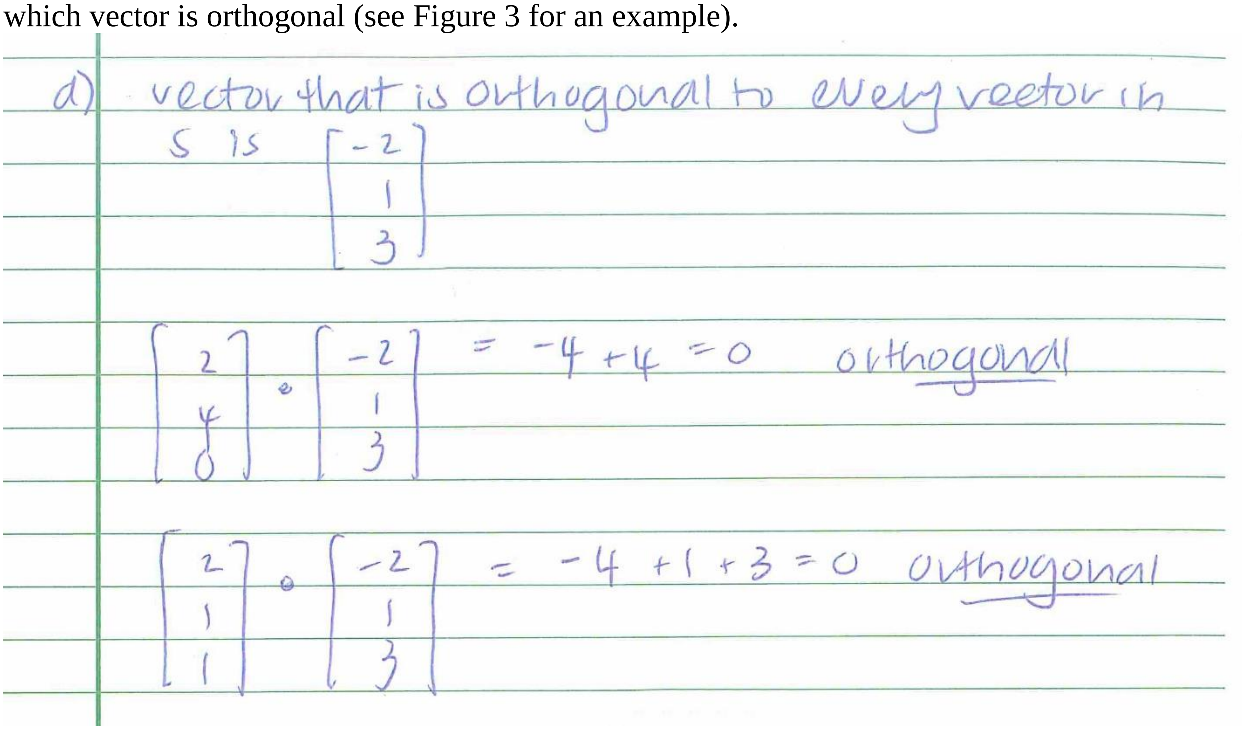

Figure 3. Example of an educated guess in Part (d).

To an external analyst who reviews students’ submissions, the described introductions of orthogonal vectors appear as an act of *guessing*. My informal conversations with seven students who submitted a written check corroborated this impression. For instance, when reflecting on her solution in Figure 2, Rina (pseudonym) said,

> This vector [in Part (d)] must be perpendicular to my vectors from the first question. So I made a system of equations first, but then I kind of guessed what vector will work. It turned out to be correct.

Rina's words resonate with Mahajan (2010), who views guessing as a valuable problem-solving approach that releases one from "the fear of making an unjustified leap" and allows her to "shoot first and ask questions later" (p. xiii). Since the checks led none of the students to the conclusion that their introduced vectors were invalid, it seems justifiable to refer to their guesses as *educated*.

In terms of Harel (2013), capturing the act of checking educated guesses in writing can be viewed as fulfilling students' intellectual needs: to ascertain the correctness of the guessed vector, to use computation as a means to show orthogonality, and for communication with the assessor, whose corresponding needs in regard to the vector should also be fulfilled. One need that this act is incapable of fulfilling is the need for causality. Indeed, guessing can be contraposed to deductive reasoning, which NCTM (1989) defines as "a careful sequences of steps with each step following logically from an assumed or previously proved statement and from previous steps" (p. 144). Many students demonstrated deductive reasoning when row-reducing matrices in Part (a), applying the standard method $A^T A\bar{x} = A^T\vec{c}$ for devising the least square solution in Part (b), using the formula $\|A\bar{x} - \vec{c}\|$ in Part (c), and stating that the vector from the second part will solve Part (d) as well.

Field-based Hypothesis

It has been repeatedly reported that students rarely bother to verify the outcomes of their mathematical doings (e.g., see Kirsten, 2018 for proving; Kontorovich, Koichu, Leikin & Berman, 2012 for problem posing; Pugalee, 2004 for problem solving). Therefore, it is notable that without being engaged in any special course of instruction, nearly a fifth of the students submitted written checks of their final answers to Part (d) in Figure 1. Some may argue that there is nothing really to notice about this as the check in this part was easier than in the other three. This argument is incommensurable with the theoretical standpoint of this paper, which operates with students' mental acts (Harel, 2008a, b) and does not ascribe cognitive properties to inanimate artefacts. Indeed, the data analysis associated students' decisions to capture their checks in writing with situations where educated guesses were involved; no checks were documented in the cases of deductive problem solving, i.e. where students operated with structured procedures, algorithms, and theorems that were taught in the course.

Within Harel's (2008a, b, 2013a, b, 2017) theory of learning, an application of a conventional procedure, algorithm, or theorem provides an apodictic epistemological justification for the emergence of a solution to a problem (see $\alpha \rightarrow \beta$ in the second section). Furthermore, in a typical learning environment, such deductive methods are purposefully promoted among students through teachers' epistemological efforts that vary in their degree of explicitness. Explicit efforts can be associated with devoting time and space to these mathematical instances during the lesson, explaining and proving them, requesting students to use them for solving problems, et cetera. More covert efforts can also be indicated. For instance, the conventional name "Gram-Schmidt orthonormalization process" promises that the process indeed orthonormalizes. At the end of the course, Rina's usage of these mathematical instances in problem solving seems inseparable from her solid belief in their high epistemological status, the one that vouches for the instances' capabilities to produce the outcomes that they were positioned as producing.

With the principle of an intellectual parsimony in mind (Koichu, 2008), it seems reasonable to propose that when Rina is convinced by a match between the assigned problem and a mathematical instance with a high epistemological status, she is unlikely to experience an intellectual need to check her solution. Indeed, the usage of the mathematical instance for devising a solution, an instance that has been actively promoted by the same authoritative figures who assigned the problem, seems "to tick many boxes" of needs, especially for certainty, causality, communication, structure, and in many cases, also for computation. If there are still doubts about the obtained solution, it seems more reasonable for Rina to review how she applied the promoted mathematics rather than to verify her final answer as a stand-alone candidate for a solution. In turn, if Rina's recollection of the mathematical instance is distorted or mismatched to the problem in hand (something that happened frequently among the participating students), it is unlikely that she will benefit from such a review.

On the other hand, as educated as guessing can be, it creates a disruption in a deductive sequence of problem-solving steps and gives birth to an outcome that comes almost "out of nothing". This disruption can not only perturb the intellectual needs of the solver but it also clashes with the usual indoctrination in a "good" mathematics classroom where no claim is accepted without being shown to be a necessary entailment. As a result, the act and the outcome of guessing can be ascribed with a low epistemological status that summons a compensation. It has been demonstrated in the previous section that this compensation can appear in the form of a special type of an epistemological justification, where a solver shows that the candidate for an answer fulfills the requirements of the assigned problem (i.e. $\alpha,\beta \nrightarrow \emptyset$).

The presented interpretation of the checking tendency that the participating students demonstrated can be framed by a chain of hypotheses as follows:

> When solving a problem, Rina can apply pieces of knowledge that she endows with different epistemological statuses. For instance, guessing and applying mathematical instances that were promoted in a classroom can be positioned at opposite ends of an epistemological scale. As a result, Rina may experience intellectual needs to check her solution that differ in terms of intensity and type. These different needs entail different checking behaviors, which predetermines to some extent Rina's chances of indicating mistakes in her own work.

Hopefully, the mathematics education community will experience these hypotheses as educated guesses that provoke a need for rigorous explorations. The potential value of this hypotheses is in linking the act of checking to the contextual affordances that emerge when a solver puts particular mathematical knowledge to use. On the theoretical level, this positioning might be viewed as an extension of previous approaches, according to which the act is driven by a solver's familiarity with verification strategies (e.g., Mashiach Eizenberg & Zaslavsky, 2004) or a matter of habits of mind (e.g., Goldenberg, 1996) that she developed. On the practical level, the hypothesis summons a search for pedagogies that are capable of provoking students' intellectual needs for checking their own solutions; the ones that are often obtained with epistemologically solid mathematics. Accordingly, I believe that explorations of the hypothesis will lead to interesting conclusions that will find their way into Rina's classroom.

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