# How Do Students Interpret Multiply Quantified Statements in Mathematics?

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*We presented introduction to proof students from five different US universities with multiply quantified statements to assess and interpret. The survey was designed to allow us to compare the influence of syntax, semantics, and pragmatics in student interpretation. We analyzed the ways students interpreted the statements both before and after instruction. Current analysis suggests that students became more sensitive to syntax (reversing quantifier order) after instruction and became better able to construct a semantically odd construal (e.g. the distance between two points is equal to multiple numbers). Our analysis of pragmatics suggests that students were more likely before instruction to construct a relevant construal, but we did not find evidence that truth-value influenced students' interpretation of the given claims.*

*Keywords*: Logic; Multiple Quantification; Introduction to Proof

Advanced mathematical language involves a number of very particular conventions of syntax and interpretation because mathematicians strive to communicate intended meanings with fidelity. Many previous studies have particularly investigated how students make sense of statements that combine universal (∀) and existential (∃) quantifiers, what we shall call multiply quantified (MQ) statements. Such statements appear quite frequently in advanced mathematics and experts almost always use them in a consistent manner, though the precise nature of the relationships conveyed varies in important ways (c.f. Durand-Guerrier & Arsac, 2005). These studies have assessed students' naïve readings of such statements (Dubinsky & Yiparaki, 2000) and have proposed and evaluated certain methods of teaching students to interpret MQ statements as mathematicians do (Dubinsky, Elterman, & Gong, 1988; Dubinsky & Yiparaki, 2000; Durand-Guerrier & Arsac, 2005; Roh & Lee, 2011). This study seeks to extend our insights into student interpretation of MQ statements by contributing a conceptual analysis of the interpretation process that is evaluated through survey instruments administered to introduction to proof students both before and after instruction. We investigate roles syntax, semantics, and pragmatics each may play in the ways students construct meaning for MQ statements.

Interpreting MQ statements in mathematics resides at the interface between mathematical logic and mathematical language. We concur with previous authors that while there exist formal rules for trying to render mathematical language purely syntactic (able to operate by precise rules ignorant of subject matter), mathematicians rarely operate in this manner and teaching novices will almost certainly require some balance between syntactic rules and semantic sense-making (Durand-Guerrier, 2003; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012).

## Conceptual Analysis and the Research Tasks

Previous studies have used the language AE ("for every-there exists") and EA ("there exists-for every") to alternatively refer to 1) the structure of a mathematical statement, 2) the normative interpretation shared among mathematicians, and 3) a student's interpretation of those statements. While we continue to use those two-letter codes for convenience, we adopt a different terminology to distinguish these constituents of the analytical process. Figure 1 presents the four statements (these are technically predicates, but we shall reserve that term for something else) that we asked students to interpret, each regarding two different referents. The *wordings of* *the statements* clearly exhibit AE or EA structure. We refer to the meaning an individual makes for any such wording as their *construal* of the statement. The construal shared among mathematicians – AE means "each to some" and EA means "one to every" – we call the *normative construal*. Each student then construes each statement in ways tantamount to "each to some," "one to every," or something else.

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|---|---|---|
| S1. "There exists a real number $M$ such that for all real numbers $x$, $f(x) < M$."<br>S2. "For all real numbers $x$, there exists a real number $M$ such that $f(x) < M$." | referents | $f(x) = 3x + 2$<br>$f(x) = \sin(x)$ |
| S3. "For every positive real number s, there exists a point $C$ on the segment [ray] such that $d(A, C) = s$."<br>S4. "There exists a point $C$ on the segment [ray] such that for every positive real numbers s, $d(A, C) = s$." | referents | segment $\overline{AB}$<br>ray $\overrightarrow{AB}$ |

*Figure 1. The four statements and four referents comprising the study tasks.*

We parse the elements that students may use to construct meaning in the following way: *quantifiers*, *predicate*, and *referent*. For instance, for S1 the quantifiers are "There exists a real number $M$ such that for all real numbers $x$," the predicate is "$f(x) < M$," and the referent is "$f(x) = 3x + 2$" or "$f(x) = \sin(x)$." To observe the influence of each, our survey alternated the order of quantifiers, the mathematical context and predicate, and the referent within each context. Students may alternatively give meaning to a statement like S1 by constructing a meaning for the syntax of quantification (one $M$ that satisfies the predicate for all $x$) or using their semantic knowledge of the boundedness property of the referent $f(x) = \sin(x)$.

To assess the role of pragmatics, we operationalized two of Grice's (1975) pragmatic maxims. Grice's maxims express rules by which interlocutors in discourse may draw reasonable implications from another's statements (possibly beyond the express meaning). We consider two: a Maxim of Quality "Try to make your contribution one that is true" (p. 46) and a Maxim of Relation "Be relevant" (p. 46). If this maxim were operative, then we would expect students to attempt to construe a false statement in some way that made it true. This maxim would be inert in interpreting a true statement. We expect this effect is preconscious, and we looked for its effect on the first statement students read in each context. The normative construal of both Statements 2 and 4 are semantically uninteresting (the former is always true and the latter is patently false), which we consider violations of the Maxim of Relation. Thus, if students avoid such a construal, this is evidence of the role of pragmatics in interpretation.

## Methodology

We designed a survey that consisted of four pairs of tasks using the MQ statements and referents in Figure 1. Each task presents a pair of MQ statements differing only by the order of quantifiers. We refer to the task as follows: S1 – EA function, S2 – AE function, S3 – AE geometry, and S4 – EA geometry. The task presentation follows for two requests for response: (1) the truth-value (true or false) of each statement for the given referent and (2) an explanation of what each statement says about the given referent.

We created two versions of the survey instrument: True-first version and False-first version. These two versions of the survey instrument contain the same tasks – four function tasks first followed by four geometry tasks – presented in different orders. For each task group (either function or geometry), T-first version presents a referent first that makes the first MQ statement in the pair to be true (EA – sine/ AE – ray), whereas F-first version presents a referent first that makes the first MQ statement in the pair to be false (EA – line/ AE – segment).

Six instructors of introduction to proof courses from five different universities in the United States allowed their students to participate to our research study in Spring 2018. We randomly assigned the student participants into two groups. To facilitate the multi-site data gathering, students completed the surveys online through an emailed link. Students were invited to complete the survey both before and after their class covered topics related to MQ statements. In this paper, we report our results from the 77 students who completed both pretest and posttest.

We first compared students' responses to the determination of the truth-value for each statement and their explanations about what each statement says. We coded a student response as EA if it exhibits "one to every" structure, AE if it exhibits "each to some" structure, and OTHER if the student construal conveyed neither such relationship or it appeared the student construed a different predicate or referent. Once we coded all student responses to each statement the tasks in terms of the three codes, we calculated how frequently students construed each statement-referent pair in a normative way. For instance, AE-sin refers to the percentage of students who interpreted S2 with reference to the sine function as an "each to some" relationship. The next section presents our preliminary analysis of these frequencies of normative construal.

## Results

Figure 2 presents the rates of normative construal by group and time. These data show two initial trends: students more frequently construed the first statement in each pair normatively and the AE sine task resulted in the lowest percentages of normative construal overall. The first pattern results in the jagged appearance of each graph. This reflects on our hypothesis about pragmatics, namely that students were less likely to construct the normative construal when its contextual meaning was either obvious (the EA function statement) or patently false (the AE geometry statement). A possible alternative explanation has to do with the order of appearance, since students always saw the more "natural" (according to normative construal) statement first.

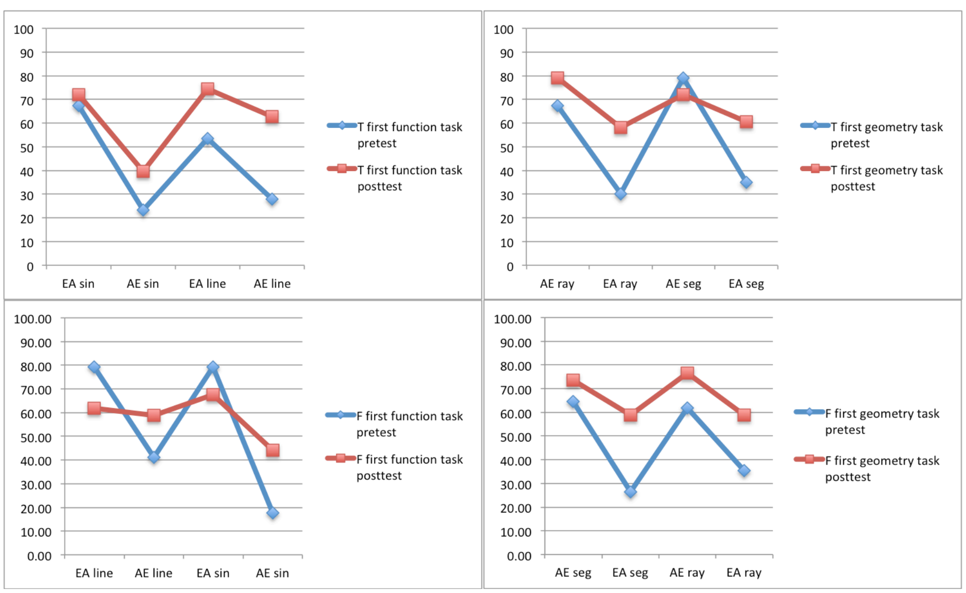


*Figure 2. Percentages of normative construal, organized by mathematical context and group.*

It may be that students construed the second statement less normatively because they had to develop a new construal for a very closely related statement, and this was more challenging. Our current study design does not allow us to fully distinguish these two explanations.

The AE sine task's low normative construal rate should be viewed in part as a byproduct of gathering data in surveys rather than interviews. While S2 entails a slightly different construal than S1 (e.g. $M$ could be .5 when $f(x) = 0$), both statements can be verified by a single $M$. Under either construal S2 is true, and students declared it so 88% of the time. When a student explains their interpretation of the AE sine task by noting that $M = 2$, this is insufficient evidence to indicate whether the student held a "one to every" or "each to some" construal. Without evidence that students thought that $M$ could vary with $x$, we did not code their responses as an "every to some" construal. It is likely that more students responded to the AE sine task according to a normative construal, but their explanation did not provide enough evidence for us to discern it. Many other explanations provided clearer evidence of either a "one to every" or an "every to one" construal, but we had to choose a system for coding ambiguous responses.

If one ignores the AE sine tasks, a third pattern arises from the data in Figure 3: instruction greatly increased the rate of normative construal for the more difficult statements (function EA and geometry AE) and resulted in a much more consistent rate of normative construal across group and context. Indeed, the rate of normative construal was above 58% (and below 80%) on all of the posttest tasks (data points marked with squares in Figure 2) except the AE sine task. We currently do not have a clear explanation for why the rate of normative construal actually decreased after instruction for some groups on some tasks.

**Influence of Task Order**

One of our primary hypotheses regarded the influence of Grice's Maxim of Quality that students might be prone to interpret statements to render them true. Operationally, would reading false statements first make students more likely to search for a (non-normative) construal that rendered the claim true? Comparing the two group's construal of each task above, the rate of normative construal differed by 10% or more on the following tasks: EA sine pre (F-First +12%), EA line pre (F-First +25.9%), AE line pre (F-First +13.3%), AE segment pre (T First +14.4%), and EA line post (T-First +12.7). Thus, the strongest evidence that the order of presentation affected student construal appeared on the function tasks prior to instruction. In this case, we see that the F-first group (who read a false statement first) were much more successful in constructing the normative construal of both EA function tasks. This suggests that in this context, reading the statement with reference to the linear function first aided students in construing the definition of bounded above with reference to both functions. This disconfirms our hypothesis that reading the definition of bounded above with reference to a bounded function would aide in developing a normative construal.

However, the geometry tasks caution against a simple explanation that seeing a false statement first is better. The T-First group fared better than the F-First group on three of the four geometry pretest tasks, with differences ranging from 5.1% to 14.4%. This means the group who saw the ray first (of which S3 is true) more frequently construed the geometry tasks normatively than did the group who saw the segment first (of which S3 is false). So, while there seemed to be some effect due to order of presentation, it varied with semantic content and not merely with the truth-value of the statement. This suggests that semantic content was more salient in student interpretation than was Grice's Maxim of Quality.

**Influence of Quantifier Order**

We assessed the influence of syntax, focused on the quantifier part of the statement, by comparing each student's construal of statements that varied only in the order of quantifiers. Figure 3 presents the percentage of students who construed corresponding EA and AE statements with the same construal. The sine task showed the greatest frequency of invariant construal at both times. Prior to instruction, students construed the other three pairs of statements the same between one third and one half of the time. After instruction, this rate dropped from between one tenth to one third of the time. Thus, reversing quantifier order frequently did not elicit a novel construal before instruction and instruction made students more sensitive to quantifier order.

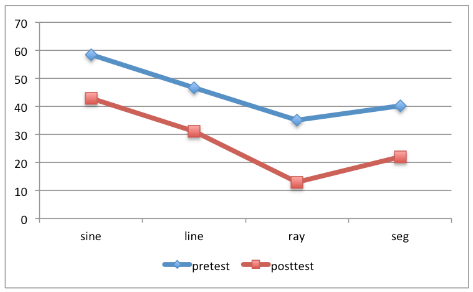


*Figure 3. Frequency of students construing different quantifier order statements the same way.*

## Discussion

This study investigates the resources that students use to give meaning to MQ statements. Our study design helped us to compare the relative roles of syntax, semantics, and pragmatics. Initial analysis of the data suggest that all three played some role in interpretation, and each aspects was at times inoperative in interpretation (for at least some students).

Syntax clearly influenced student interpretation, inasmuch as students normatively construed the various statements with at least modest frequency, especially after instruction. However, before instruction students also constructed the same construal for AE and EA statements at least 30% of the time. Semantics clearly played a role inasmuch as the pattern of interpretation was quite different between function and geometry settings. As was expected in the study design, it appeared that "one to every" relations were easier to construct in the function context and "each to some" relations were easier to construe in the geometry context. Instruction seemed to shift interpretation from semantics toward syntax inasmuch as the posttest rates of normative construal were less varied.

Regarding pragmatics, we did not find support for the claim that Grice's Maxim of Quality influenced students' interpretations. On the function items, students fared better when they first read the definition of bounded above with reference to an unbounded function. A possible explanation is that since they could not give meaning to the statement based on their understanding of the sine function's prominent property of being bounded, they had to attend more closely to the quantifier structure. The data may support the role of the Maxim of Relation in explaining why certain statements were uniformly harder to construe normatively, but we cannot rule out that order of appearance explains this pattern instead.

Ongoing analysis will attend to other details of student construal such as how explicitly they explained quantification and the dependence between variables. We also plan to conduct statistical analysis on the data presented here. In our presentation, we will discuss the following:

1. How can we explain the reduced rate of normative construal on some tasks?
2. What other comparisons and analyses should we conduct on the data?

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