Exploring College Geometry Students' Understandings of Taxicab Geometry

Jose Saul Barbosa
University of Texas at San Antonio

Priya V. Prasad
University of Texas at San Antonio

Non-Euclidean geometries are commonly used in college geometry courses to highlight aspects of Euclidean geometry. Scholars have theorized that working in non-Euclidean geometries requires thinking at the highest van Hiele level of geometric thinking, which was developed by investigating students' learning of Euclidean geometry, but few have pursued this empirically. This empirical study seeks to develop levels of geometric thinking for students in Taxicab geometry, which is the non-Euclidean geometry that is closest in structure to Euclidean geometry. Students in a college geometry course that included prospective secondary teachers were audio-recorded in group discussions as they completed tasks about congruence and transformations in taxicab geometry, and their written work was collected. Portraits of participants' thinking about Taxicab geometry were developed, leading to a proposed structure for the levels of geometric thinking for Taxicab geometry.

Keywords: College Geometry, Student Thinking, Preservice Teachers

Introduction

College geometry courses can include a wide variety of students, from mathematics majors to preservice middle-grades and secondary teachers. These courses commonly introduce some form of non-Euclidean geometry, either as a worthwhile topic in its own right or as a way to highlight subtleties or assumptions in Euclidean geometry. The prevalence of non-Euclidean geometry in these courses makes student thinking about non-Euclidean geometries a useful topic for exploration in research. The van Hiele model was developed to characterize the different levels of geometric understanding and can be used to assess students' levels of geometric thinking. The van Hiele model has generally been researched with applications to Euclidian geometry, but non-Euclidean geometries may also be introduced to students in a college geometry course. However, limited research has been done on how the van Hiele model can be applied to non-Euclidean geometries. A non-Euclidian geometry that can be found in the curriculum for an undergraduate geometry course is Taxicab geometry. Due to its inclusion in curriculum, we conducted a preliminary exploration of the levels of thinking in Taxicab geometry.

Background

The van Hiele Levels

The van Hiele levels of geometric thought are a way of identifying a student's level of geometric thinking (Crowley, 1987). The van Hiele levels are as follows:

1. *Visualization:* This is sometimes considered the base level. In this level, students can name figures judging by their appearance, but their properties are not understood.
2. *Analysis:* In this level, figures are bearers of their properties and students can reason that they are classified based on their properties. However, the properties have no logical order to them.
3. *Informal Deduction:* In this level, students can deduce that one property precedes or follows another property. Definitions are introduced. Students are also able to give informal arguments to justify their statements and follow formal proofs, but cannot construct a formal proof from a different or unfamiliar premise.

4. *Deduction:* Students can now construct proofs and see relationships between definitions, axioms, and theorems and use them to establish further theory and can distinguish between statements and their converses.
5. *Rigor:* Geometry can now be seen abstractly by students. The students can work with different axiomatic systems to further study non-Euclidean geometries.
(van Hiele, 1959/2004; Crowley, 1987)

One of the properties that accompanies the levels is the *sequential* property which states that to achieve one level, one must have achieved all prior levels too (van Hiele, 1959/2004; Crowley, 1987). Mayberry (1983) displayed that the sequential property did take effect on her study of the van Hiele levels. Research conducted on college students demonstrates that most students only achieved level 3 thinking during a college geometry course (Mayberry, 1983; Wang, 2011). Research suggests that if students do not progress through to the fourth level, then they are not likely to succeed in a college based geometry course where they are generally expected to reason deductively (Mayberry, 1983; Wang, 2011).

Taxicab Geometry

Students in college geometry courses are most likely to have been exposed to Euclidean geometry in school, and most prospective teachers are only required to teach Euclidean geometry in their future careers. Other geometries can be constructed through changes in the axioms or the metric of Euclidean geometry. Taxicab geometry is created by changing the Euclidean metric to the Taxicab metric. In Euclidean geometry, the distance between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is measured by the length of the straight line connecting the two points (analytically, $d_E(A, B) := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$) while Taxicab geometry measures distance by taking the sum of the horizontal and vertical distances between the two points ($d_T(A, B) := |x_2 - x_1| + |y_2 - y_1|$). This is demonstrated in Figure 1. A common analogy that is used pedagogically to describe this new metric is of a taxicab driving in a grid-like city such as Manhattan (Krause, 1986).

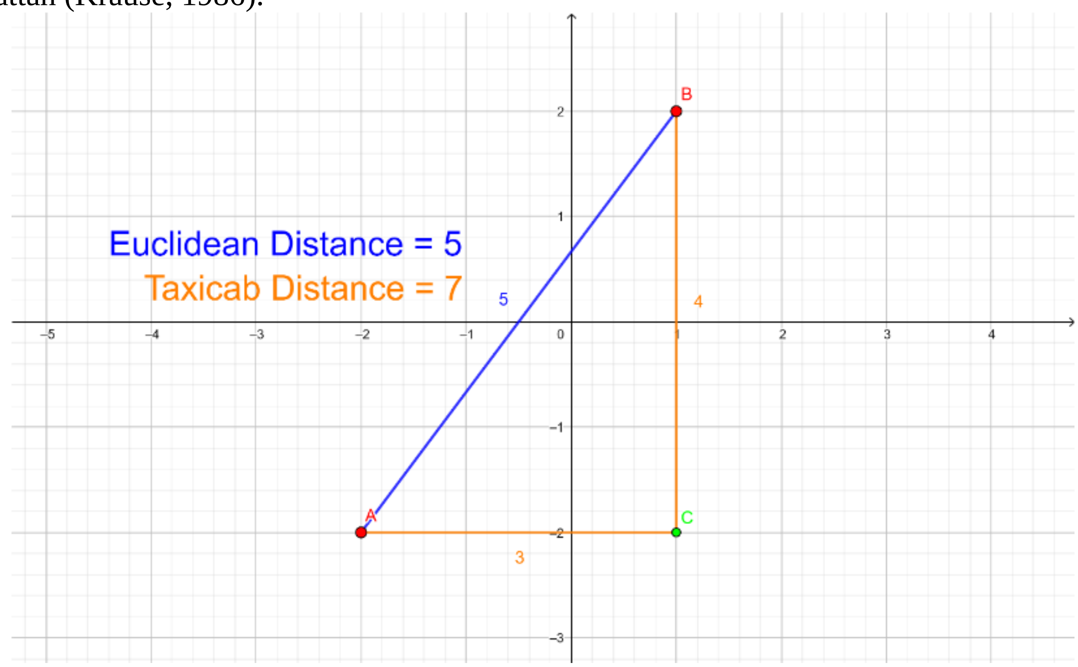

Figure 1: Euclidean distance vs. Taxicab distance.

Research on Understandings of Non-Euclidean Geometries

There is a lack of research concerning how levels of understanding, like the van Hiele levels for Euclidean geometry, would look for non-Euclidean geometries. Kemp & Vidakovic (2017) mention that despite over two decades of research on the van Hiele model, few participants have been classified to be at the fifth level of geometric thinking, which could be why the Rigor level has not been studied, resulting in a lack of research in levels of understanding for non-Euclidean geometries. While there is limited research on how levels of understanding in non-Euclidean geometries would be described, Guven and Baki (2010) have developed levels of understanding in spherical geometry known as Transition, Definition Comparison, Pre-Deductive, and Deductive, which develop sequentially. This work was very influential to this particular study.

Research Question

Since Taxicab geometry has the potential to improve the understanding students have of Euclidean geometry, particularly with respect to ideas about congruence (Boyce & Prasad, 2018), it becomes useful to understand how students develop their thinking about Taxicab geometry. This study posits a preliminary results in response to the following question: What are students' levels of geometric thinking in Taxicab geometry?

Methods

The setting for the data that was collected for this study was a college geometry course at a large southwestern university. Prospective middle and high school teachers are required to take this course and made up around 50% of the class. A course in introduction to proof writing was not a prerequisite. Students participated in a week-long exploration of Taxicab geometry. After the students were introduced to the Taxicab metric, they studied relationships between Euclidean and Taxicab geometry. The following tasks were assigned to the students in the course to gather information of how they understand Taxicab geometry.

1. Come up with examples of each, or explain why such an example is not possible
    a. 2 triangles that are congruent in both Euclidean geometry and Taxicab geometry
    b. 2 triangles that are congruent in Euclidean geometry but **not** Taxicab geometry
    c. 2 triangles that are congruent in Taxicab geometry but **not** Euclidean geometry
2. Identify all the Taxicab isometries.
3. How can we define congruence in Taxicab geometry?

Students worked on these problems in assigned groups of roughly four students; these groups were audio recorded and their written work was also digitized and synced with the audio recording using LiveScribe dot paper and smart pens ("Dot Paper", n.d.). The researcher took notes over all group recordings to search for key pieces of dialogue and writing in order to identify group recordings for transcription. The authors transcribed the aforementioned groups' recordings and took notes to find recurring patterns in the thought process of the students. Using these patterns and following Guven and Baki's (2010) levels of understanding in spherical geometry and van Hiele's levels of geometric thinking (van Hiele, 1959/2004), the authors created preliminary levels of geometric thinking for Taxicab geometry.

Results

Students' group work generally followed a few particular trajectories of thought, prompted by these tasks. These helped the authors propose the following preliminary levels of thinking in the Taxicab geometry.

Level 1: Transition

The student is now aware that they are studying a geometry different to Euclidean geometry because of the difference in metric. The student names observed figure in a manner consistent with Euclidean geometry. Most groups displayed this when they answered the first bullet point of the first task, spending time discussing the manner in which to measure the length of the hypotenuse in a right triangle (see Figure 2).

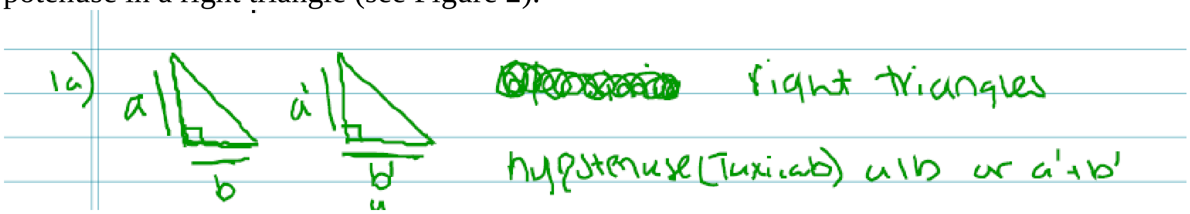

Figure 2: Students' work as they discuss the change in metric.

Level 2: Geometry Comparison

The student can focus on definitions of concepts and figures and learns to represent them visually on the Cartesian plane and can compare what they are and do in either geometry. While the student understands what the concepts are, they do not know how to use them for problem solving. For example, students demonstrate that the orientation of a figure will affect distance on that figure. The group whose work is shown in Figure 3 changes the orientation of a triangle so that the sides would be calculated differently, comparing in either geometry after changing the orientation.

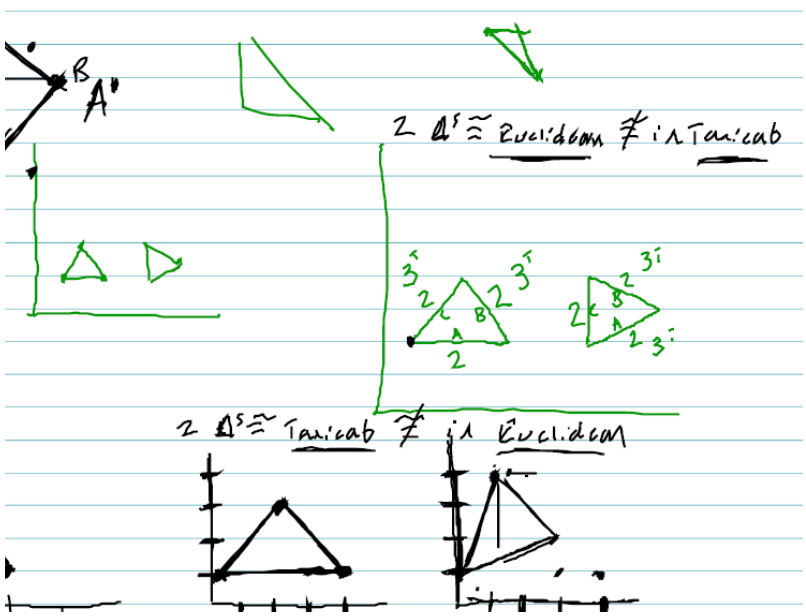

Figure 3: Students' exploration of how orientation affects distance.

Level 3: Pre-Deductive

The student can solve problems using Taxicab geometry constructions but does not yet solve problems involving deductive reasoning. The student can follow formal proofs but cannot alter the logical order of the proof or deduce a new proof. Only one of the groups studied attempted to use a Taxicab circle to justify a transformation where they placed one point of a triangle that was on a Taxicab circle to another point on that circle. They made the assumption that this would preserve distance since the distance from the center would not change but they did not account for the side of the triangle that did not share a point with the center of the circle.

Discussion

As expected by the Sequential property that both the van Hiele levels of geometric thinking and the levels of understanding in spherical geometry share, there were many examples of Level 1 thinking throughout the student discussion while they worked on their assignments. Naturally, due to this being an early exploration, there was much discussion about the Taxicab metric. In discussing the second level of thinking in Taxicab geometry, a recent study by Kemp and Vidakovic (2017) shows a student having an understanding of the definition of a circle and recognition that the Taxicab circle will appear in a different manner to the traditional Euclidean circle but fails to construct such Taxicab circle. This situation shows growing thinking in our proposed second level of thinking in Taxicab geometry, Geometric Comparison, since the student was aware of the definition of the circle and showed development in the visualization of the Taxicab circle. The group data collected in this research displayed the groups using the notion of orientation to affect distance in Taxicab geometry. This gives insight to their understanding of the rotation transformation and the effect it had in Euclidean geometry. Only one group contributed to the development of Level 3. Due to the students studying Taxicab geometry for only a week, it was not expected that there would be many results to provide further analysis of this level. While they were not completely correct, this type of work helped highlight the characteristics of our proposed third level.

The results in this paper do not include a Deduction level as previous findings do. This is primarily because none of the groups displayed characteristics expected of a student that can make deductions in Taxicab geometry. However, based on the Deduction levels proposed by van Hiele (1959/2004) and Guven and Baki (2010), the following was hypothesized as a candidate of the fourth level of thinking in Taxicab geometry (also called Deductive): The student can prove propositions deductively and support theorems through more universal definitions rather than specific geometry definitions. The student can make deductions from such definitions that are universal to other geometries. A hypothetical example of a student thinking at this level of would be able to give a proof or a counterexample to the following statement, "All rotations by $\frac{k\pi}{2}$ are isometries in both Euclidean and Taxicab geometry. $k \in \mathbf{Z}$."

Next Steps

The assigned tasks used to gather data for this research were not designed with the intent of developing levels of thinking in Taxicab geometry; thus, the levels proposed here are preliminary. Future plans for this research include conducting individual interviews in order to get a better understanding of the characteristics that can form the levels of understanding in Taxicab geometry, as Guven and Baki (2010) did to develop levels of thinking in Spherical geometry.

References

Boyce, S. & Prasad, P.V. (2018). Supporting prospective teachers' understanding of triangle congruence criteria. In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, California, 139-153.

Crowley, M. (1987). The van Hiele Model of the Development of Geometric Thought. *Learning and Teaching Mathematics*.

Dot Paper. (n.d.). Retrieved June 12, 2018, from http://www.livescribe.com/en-us/support/ls3/dot_paper.html

Guven, B., & Baki, A. (2010). Characterizing student mathematics teachers' levels of understanding in spherical geometry. *International Journal of Mathematical Education in Science and Technology, 41*(8), 991-1013. doi:10.1080/0020739x.2010.500692

Kemp, A. & Vidakovic, D. (2017). A Student's Use of Definitions in the Derivation of the Taxicab Circle Equation In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education, San Diego, California.

Krause, E. F. (1986). *Taxicab geometry: An adventure in non-Euclidean geometry*. New York: Dover Publications.

Mayberry, J. (1983). The Van Hiele Levels of Geometric Thought in Undergraduate Preservice Teachers. *Journal for Research in Mathematics*.14, (1), pp 58-69

van Hiele, P.M. (1959/2004). The Child's Thought and Geometry. In T.P. Carpenter, J. A. Dossey, & J.L. Koehler, (Eds.). *Classics in mathematics education research*. National Council of Teachers of Mathematics.

Wang, S. (2011).*The Van Hiele Theory through the Discursive Lens: Prospective Teachers' Geometric Discourses* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global. (919077177)