Providing Undergraduates an Authentic Perspective on Mathematical Meaning-making: A focus on Mathematical Text Types

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A disciplinary literacy perspective suggests that the goal of instruction in any discipline is to apprentice students into increasing participation in the disciplinary community. In this paper we explore four distinct types of mathematical texts and the critical role each plays in mathematical meaning-making. We argue that understanding the nature and uses of mathematical text types moves undergraduate students closer to the goal of approximating/engaging in mathematical practices, resulting in greater access to powerful mathematics.

Keywords: mathematical practices, disciplinary literacy, reasoning and proof, algebra and algebraic thinking

Mathematics with Purpose

When mathematics is used in the world it is always *for a purpose*. An engineer writes mathematical text in order to make structures more resilient to the elements (Grayson, Pang, & Schiff, 2012). A computer scientist writes mathematical text in order to predict the next unrest event (Dopson, Lowery, and Joshi, 2014), while an applied mathematician writes mathematical text in order to create better tools for image compression (Roach, 2010). Therefore, understanding the nature and uses of mathematical text types moves undergraduate students closer to the goal of approximating/engaging in mathematical practices that make use of text in varied ways.

A Disciplinary Literacy Perspective on Uses of Mathematical Texts

A disciplinary literacy perspective suggests that the goal of instruction in any discipline is to apprentice students into increasing participation in the disciplinary community. That is, to engage students in the practices of those who generate, critique and disseminate knowledge in a given field. As Gabriel & Wenz (2017) note, members of a disciplinary community utilize "agreed-upon conventions that guide the production, communication, and critique of disciplinary knowledge. The central goal of disciplinary literacy instruction is to help adolescents develop 'insider status' in these communities." We began our inquiry by working to uncover the nature of texts used in mathematics by interviewing six pure and applied mathematicians across the US to explore the ways in which members of the field of mathematics orient to and engage with texts.

It should be noted that the notion of mathematical text types is distinct from multiple representations, in that utilizing multiple representations is an essentially pedagogical technique in which showing something multiple ways is intended to promote student understanding. In contrast, within the discipline of mathematics, one would usually not create multiple representations of a single idea without a rationale. Instead, each mathematical text type is utilized with a specific mathematical purpose in mind.

Mathematical Text Types

Within disciplinary literacy circles, the phrase "mathematical text" often conjures up one of two visions: a math textbook (Feng and Schleppegrell, 2010; Shanahan & Shanahan, 2008) or a mathematical proof (Moje, 2007; Moje, 2008). With respect to the former, we agree with Fang and Coatoam (2013) that "school subjects are disciplinary discourses recontextualized for educational purposes" (p. 628). This is not to discount the important and thoughtful pedagogical work that goes into creating a school textbook. Our argument is that the texts students interact with most often are not necessarily representative of text types used in the doing of mathematics outside of classroom settings (applied or pure mathematics). As a pedagogical text, its main purpose is to instruct, and its main audience is outsiders to the discipline's community. This is analogous to the way in which biology students study frogs. When a frog is recontextualized for educational purposes, the resulting corpse is more easily studied, yet lacks much of its inherent frog-ness (hopping, eating flies, croaking, etc.) that made it of interest in the first place. In a mathematics textbook, mathematical ideas are explained and demonstrated, but perhaps not communicated in ways that are authentic to the discipline's everyday work of generating, critiquing and sharing knowledge.

Mathematical proof, on the other hand, is an example of an authentic mathematical text written to convince the reader of a mathematical claim (existence proof), or explain why that claim is true (constructive proof). In fact, this type of text is so synonymous with the discipline of mathematics that some literacy researchers view it as *the only* type of mathematical text (Moje, 2007; Moje, 2008). In our view, to limit mathematical text to only proof text would be to further privilege pure mathematics over applied fields of mathematics, fields which rely much more heavily on the other three text types (algebraic/symbolic, algorithmic, and visual).

In the following paragraphs, we briefly introduce the other text types, providing not only a description of each, but also supporting evidence for the text types from a diverse set of fields including linguistics (O'Halloran, 2005, 2015; Pimm, 1987), literacy (Draper and Broomhead, 2010; Feng and Schleppegrell, 2010; Moje, 2007; Shanahan & Shanahan, 2008), history of mathematics (Cajori, 1993; Maur, 2014), and mathematics education (Kaput, Blanton, and Moreno-Armella, 2008). For each text type we highlight the purpose and the text features which are used to accomplish that purpose.

Technologically Driven, Algebraic/Symbolic Text

The purposes of algebraic/symbolic texts are to *generalize* and *condense* (see Kaput, Blanton, and Moreno-Armella, 2008). The key feature of algebraic/symbolic text is specialized notation (Pimm, 2015).

Algebraic/symbolic texts have developed as a natural outgrowth of mathematical work, as necessary tools for mathematical meaning-making, especially when it comes to increased levels of abstraction (for full descriptions of the history of mathematical symbols, see Cajori (1993) or the more recent and less terse work of Mazur (2014)). Since the early Renaissance, developments in mathematics are inextricably linked to developments in mathematical writing and associated technologies. For example, algebra can be thought to have developed in three stages: rhetorical, syncopated, and symbolic (O'Halloran, 2005). Rhetorical algebra is mathematics in which unknown quantities are referred to using words instead of symbols. As the printing press increased access to mathematical texts, mathematicians began to develop new arithmetic algorithms, while simultaneously standardizing mathematical procedures and symbols. This would set mathematics on a path to syncopated algebra (a combination of words and symbols), and finally symbolic algebra, with the first algebra text *Summa de Arithmetica* printed in 1494. (O'Halloran, 2005). By the time of Descartes, symbols were able to "liberate algebra from the informality of the word" (Mazur, 2014, p.xvii).

Notice how algebraic text allows us to treat complex relationships as a single object (or collection of objects). In order to work with algebraic text in any meaningful way, we must be able to unpack that complexity at will, attending to aspects of form relevant to the mathematical task at hand. Undergraduate students should not be expected to read mathematical text in this way as a simple byproduct of engaging in mathematics (Ferrari, 2004). The cultivation of such disciplinary habits of mind requires explicit instruction. Moreover, attending to the historical development of symbolic texts is important since "causes of the success or failure of past notations may enable us to predict with greater certainty the fate of new symbols which may seem to be required, as the subject gains further development" (Cajori, 1993, p.196).

Mathematical Processes, Algorithmic Texts

The purpose of algorithmic texts is to provide access to problems that have *no known analytic solutions* or are *too large in scale* for hand computation (e.g., large datasets or a large number of cases to consider). The key features of algorithmic texts are control structures.

Along with symbolic text, algorithmic text is likely one of the two oldest text types, occurring anywhere that a person needed to perform a mathematical procedure repeatedly. While today we often associate algorithmic text with computer code, computers are a sufficient but not necessary condition for the use of these texts. For example, the classic description of how to approximate the square root of a number by repeatedly averaging successive guesses (Joseph, 2010) is an algorithmic text that can be read and implemented on paper. Entire fields of mathematics now rely heavily on algorithmic text, including numerical analysis, numerical linear algebra, discrete mathematics, and computational algebraic topology to name a few.

From Cannon Balls to Pendulums, Visual Text

The purposes of visual text are to *highlight relationships* and *appeal to mathematical intuition*. The key features of visual text are static aspects of a functional relationship such as domain, range, maxima, or minima, or more dynamic aspects, including end behavior or average rate of change.

Visual texts "enable mathematicians to represent the linguistically and symbolically encoded information in ways that are tangible to the human perceptual sense" (Fang, 2012, p.26). Historically, visual text has been spurred on by the advent of new technologies. Initially, advances in printing allowed for diagrams to be included in mathematical texts. The first of such texts in western mathematics included diagrams as *additions to* the mathematics being discussed (O'Halloran, 2005). But, in the hands of Newton and Leibniz, visual text *became the mathematics.* For example, the method of integration by parts undergraduates learn in second-semester calculus is based on visual text and an accompanying geometric proof which relates the distance between an axis and the geometric center of a figure (i.e., the moment of the figure) (Suzuki, 2002). In fact, the calculus of Newton and Leibniz is largely the study of curves and the various types of change they encompass (including slope of the tangent line to a curve and area under a curve). More recently, since the 1980s, computers have allowed for the further integration of visual text into mathematical work, both inside and outside of the classroom. Notice that, by our definition, everything from graphs of functions, to tables of values, to interactive diagrams in DESMOS all count as visual texts. For the current discussion we focus on graphs of functional relationships due to their prominence within the undergraduate curriculum.

Feedback from disciplinary insiders: Initial results from a Delphi study. In order to further refine our understanding of the nature and use of these mathematical text types, we are currently undertaking a modified Delphi study (Green, 2014) of research mathematicians from several universities, representing multiple mathematical subdisciplines. While a complete discussion is beyond the scope of this paper, we present preliminary results from the second round of our modified Delphi, a focus group of 6 current research mathematicians (2 pure and 4 applied), representing 4 different colleges and universities from the southeast, and 5 subdisciplines of mathematics. In the following paragraphs we highlight specific facets of the text types that these researchers were able to highlight, providing us with new insights into how these types of texts are created and negotiated within the discipline of mathematics.

Visual Text: "...Not just a Cheap Cartoon Version of Proof"

Defining the role and legitimacy of visual text as a means of mathematical meaning making has been a long running debate within mathematical circles. Davis (1974) provides one of the more ardent defenses of visual text, with his notion of "mathematical theorems of perceived type":

> The analytic program [algebraic/symbolic text], then, is a prosthetic device, acting as a surrogate for the 'real thing.' The unit circle as perceived by the eye and acted on by the brain is a very different thing from the symbol string $x^2 + y^2 = 1$...The visual circle is the carrier of an unlimited number of theorems which are instantly perceived. (p. 119)

Algebraic text and its inherent malleability allow for the proof of mathematical results in ways that are not as readily possible with the often static visual text (O'Halloran, 2005). But for applied mathematicians who often create graphs *as* a mathematical result, visual text can play a much more central role in mathematical meaning making, a perspective which naturally bleeds over into their classroom teaching.

> *Participant*: In my world of applied math I try to get students to realize that that graph might be the answer. The whole solution text *is* that visualization. That's not just a cheap cartoon version of a proof. It is itself in fact a mathematical object. Now I don't know if I have convinced a lot of people of that…Everything else is secondary to proof, I know that that's the way it is, but I don't think I would teach like that. I don't teach like that.

This mathematician is aware of "the way it is" (visual text as secondary within the wider disciplinary community) and chooses to push back against prevailing disciplinary norms by providing an alternate perspective to his students. The separation between texts used by and for students and those used by mathematicians was a frequent theme across the interview. This signals a distance between the textual practices of math students and those used by insiders in the disciplinary community. Advocacy for the use of visual text with students was also echoed by another mathematician:

> *Participant*: Visual text is by far the most powerful. It's what I use in my classes. I use it instead of proof, and it seemed ok, but there is a part of me that's thinking I'm only representing an example, I can't draw all graphs that are decreasing or increasing. But, I can show them one graph that is decreasing, and sure enough, the tangents are all negative. I mean, it feels like a little bit of a cheat, but by far the students get it. Whereas,

you do the algebraic/symbolic text, they might not get it, or it might not be as meaningful to them.

The notion of visual text as pedagogically powerful and meaningful was a recurring theme. Not only did mathematicians have this view when working with their students, students had this view when thinking forward to their future interactions in the workplace.

> *Participant*: I had a conversation about that point just the other day with my students in an applied math class. We're doing Fourier series in two variables. You solve a problem and there's the Fourier series. I asked them, and they were essentially all engineers, would you present this to your boss if your boss gave you that problem? Everybody said "Well God no! The boss would fire me!" Well what would you do? And the thoughtful students said "Well I'd show the surface. You know, I'd show the picture of the surface to the boss. That would give him the information he wants. That's the sort of text, it's a visual sort of text, not an algebraic text."

Again, this underscores the importance of purposeful choice of mathematical text type. When you want to convey mathematics with meaning, either from expert to novice or across disciplinary boundaries, you'll likely choose visual text. Understanding how and when that communication is necessary is part of the work of engaging in mathematics purposefully or using it to do work outside of a school setting. Moreover, understanding the implications of certain choices for different audiences can contribute both to students' understanding of mathematical texts types and their decisions to make use of one or another. The quotes above also demonstrate the orientation towards different text types within the discipline - with an implied hierarchy of purity and legitimacy on the one hand, and a hierarchy of utility and accessibility on the other. Unlike other disciplines where the use of a certain text type and purpose for writing may be more neutral or may be defined entirely in relation to a certain sub-discipline (e.g., nonfiction in journalism), decisions about text type in mathematics convey something about the identity and purpose of the creator and their intended audience.

Algorithmic Text: "...What is a Number?"

Initially, algorithmic text made the list of text types as a clear rebuttal of the notion that proof text was the only type of mathematical text. During the focus group interview our participants discussed the ubiquity of algorithmic text across all fields of mathematics. For example, one participant underscored the strong mathematical relationships that tie together numbers and algorithms.

> *Participant*: One thing we deal with in numerical analysis is "what is a number?" I mean, do real numbers even exist, ones that can't be represented in a computer, that can't be constructed? So even like $\sqrt{2}$. That is a nonterminating decimal. Now it is the diagonal of a square with sides one by one. But what *is* $\sqrt{2}$? …one argument we do talk about in my field is numbers are things that can be represented either as the conclusion of an algorithm, like Newton's method, or numbers that are the limit point of something that can be described.

These comments helped the authors to better understand algorithmic text as it is viewed within a community of mathematicians. To mathematicians, algorithmic texts are not simply code on a computer, but also include any well-defined process that terminates. As a result, algorithmic texts are an essential part of both pure and applied mathematics that facilitate certain kinds of mathematical processes. When students are introduced to these mathematical texts, the mathematical purpose (not just the communicative features) should be conveyed, so that students learn to make use of mathematical text *in the work of mathematics* rather than viewing them only as a way to capture or record mathematics after the fact. In other words, knowing what such texts can be used to do in the processes of mathematical thinking interrupts the idea that mathematical texts are created as an end product for others - an idea that is reinforced if the main exposure to mathematical texts is within a textbook. Students should be aware that mathematical texts have a key role in facilitating the process of doing of mathematics so that they can engage with them in this way.

Proof Text: **"...Mathematics is so Much Bigger Than That Now"**

Proof text has traditionally been viewed as the pinnacle of mathematical rigor and achievement, by both disciplinary insiders and the layman alike. Such a perspective can often be at odds with how mathematicians go about their daily work, especially for those in more applied fields.

> *Participant*: It is tough. I think proof has a privileged status in mathematics. As a numerical analyst myself…there are things you can't prove. The best wavelet compression for a fingerprint? It depends on the fingerprint! So, some things you cannot prove. But there is definitely a bias towards things that are not followed up with a proof or knowing when this formula works, that's important in mathematics.

Such bias has direct implications not only for practicing mathematicians, but also for the next generation of scholars.

> *Participant*: I just advised a PhD student and there wasn't a single proof in his dissertation, right? And those papers are getting published, applied papers about models and results, numerical algorithms, optimization, parameter estimation…so, I think it's a much more broad spectrum of things that are published and acceptable.

This participant is making the case that the mathematics she and her students create, mathematics without proof text, ought to be viewed as disciplinarily "acceptable," where here acceptable is synonymous with mathematical rigor. The argument would be that as the discipline of mathematics changes, so too must the nature and role mathematical texts in mathematical meaning making. This is nowhere more apparent than in the increasingly interdisciplinary nature of mathematics itself.

> *Participant*: And all the connections that mathematics has made, I think biology has been a big driver in terms of change. All the sciences and social sciences have expanded what is meaningful, right? A biologist doesn't care about a proof, necessarily, right?

While the discipline may be ever-changing, the perspective on the discipline that is enacted in the K-12 classroom is highly resistant to change. As a result, proof text can act as a gatekeeper to future mathematical engagement, providing an “inauthentic” perspective on what it means to do mathematics.

> *Participant*: I would say an undergraduate major doesn’t really fully understand the concept of proof and able to produce at any kind of maximal level, maybe junior or senior and I think there are students who not until they are a graduate student. To be pushing that into the high school, and holding that up as a standard of achievement? I mean most students are not mathematically mature, and to say to them “you are not going to be a mathematician,” or whatever you are saying, to a 10th grader who can’t do proofs in geometry? If you say: this is what a mathematician does. You can’t do it. Therefore, this path is not available to you? Inauthentic! It’s so narrowly defining mathematics that you are eliminating the option for so many people. Because mathematics is so much bigger than that now.

Thus, proof writing is only one of several disciplinary practices used by those who engage with mathematics. Moreover, it is a sufficient, but not necessary practice for full engagement with the discipline. In contrast, disciplinary literacy practices play a fundamental role in the work of today’s mathematicians (both applied and pure), and are a necessary for any real type of mathematical meaning making. As a result, they represent a high leverage opportunity for teachers.

Conclusion

Discussion with our participants made us aware of notions of access to powerful mathematics that we would not have previously associated with mathematical text types. This further underscores the multifaceted role that mathematical text types play regarding not only the enactment of mathematical practices, but also issues of equity, including “students’ development of a sense of efficacy (empowerment) in mathematics together with the desire and capability to learn more about mathematics when the opportunity arises” (Cobb and Hodge, 2010, p.181).

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