Mathematical Errors When Teaching: A Case of Secondary Mathematics Prospective Teachers' Early Field Experiences

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The construct of mathematical knowledge for teaching (MKT) has transformed research and practice regarding the mathematical preparation of future teachers. However, the measures used to assess MKT are largely written tasks, which may fail to adequately represent the nature of content knowledge as it is used in instructional decision making. This preliminary report shares initial findings into one measure of MKT in practice – mathematical errors made during planning and enactment of mathematics instruction. We analyzed lesson plans and classroom video from prospective secondary mathematics teachers (PSTs)' supervised field experiences in college algebra course. We found that there tended be more errors related to understanding of functions (especially logarithmic), but relatively few errors happened overall during instruction. Of the errors made during planning, the majority of these errors were issues of mathematical precision. Implications for the mathematical preparation of secondary PSTs, as well as research on MKT in practice, are discussed.

Keywords: Pre-Service Teachers, Mathematical Knowledge for Teaching, Content Knowledge, Early Field Experience, Secondary Teacher Preparation

Introduction

Although the knowledge of mathematics teachers has been a widely discussed and researched topic for decades, surprisingly little empirical research has examined evidence of teachers' mathematical knowledge from teaching episodes. The mathematical errors teachers make during instruction, particularly when consistent, may reveal aspects of their content knowledge that need further development. Certainly, anyone who has taught mathematics knows that making mathematical errors, when unintentional, is inevitable. Yet, surprisingly little empirical evidence, especially when compared against the extensive research on students' mathematical errors, exists regarding the nature of mathematical errors made by teachers during mathematics instruction. Such work could shed light on the robustness of novice teachers' content knowledge as they engage in the complex decision making inherent to classroom teaching, and suggest areas where novice teachers' mathematical knowledge might be further developed prior to completing teacher preparation.

In this paper, we present exploratory research extending existing work measuring the nature of teachers' content and pedagogical content knowledge using written assessments (e.g., Hill, Schilling & Ball, 2004; McCrory, Floden, Ferrini-Mundy, Reckase & Senk, 2012) that investigates the kinds of mathematical errors secondary PSTs make when planning and teaching mathematics. The results suggest not only interesting directions for future research on PSTs enactment of their content knowledge while teaching, but also implications for content and methods courses in terms of topics where PSTs may need reinforcement of their knowledge prior to being certified and, more importantly, how to support PSTs in managing moments where the inevitable mathematical errors will happen.

Theoretical Framework

Much of the contemporary work in teacher education is founded upon the assumption, which some research has established empirically (Wilson, Floden & Ferrini-Mundy, 2002; Hill, Umland, Litke & Kapitula, 2012) that teachers' knowledge influences their teaching practice. As a result, a number of projects to improve novice secondary mathematics teachers' practice have aimed to develop prospective teachers' content knowledge for teaching (Garet et al., 2016; Sevis, Cross & Hudson, 2017). However, much of the existing empirical research to understand and measure teachers' content knowledge for teaching have involved the use of specifically designed written tasks rather than attending to how knowledge is used during practice. While written measures are certainly easier to implement and analyze at a large scale, they are imperfect measures of how a teacher might use or draw upon their content knowledge during instruction (see Shechtman, Rochelle, Haertel & Knudsen, 2010). Through an analysis of secondary PSTs' planning and enactment of instruction in an early field experience, the research question addressed by this study was: *What characterizes the kinds of mathematical errors made by novice secondary mathematics teachers when planning and enacting mathematics instruction*?

Methods

To address the study goal, we analyzed data collected as part of a larger study to investigate the opportunities to learn about mathematics teaching through an early field experience planning and teaching lessons in a college algebra course. This experience was a required component of a secondary mathematics methods course participants were concurrently enrolled in. All participants were in their senior year of a 5-year, university-based, secondary mathematics teacher preparation program, which requires candidates to complete a Bachelor of Science degree in Mathematics, along with education coursework and a full-year student teaching placement in their fifth year of the program. A total of 14 PSTs (n=14) agreed to allow members of the project team to analyze the videos of their teaching in the college algebra class, as well as analyze their lesson plan artifacts (mathematics pre-planning worksheet (P1), initial lesson plan (P2), and revised lesson plan (P3)).

To code the enacted lessons for mathematical errors, we first assembled the collection of instances where mathematical errors had occurred as captured on video of the 14 lessons taught by pairs of PSTs (each pair taught a lesson twice in the course). The first step to building this collection was to isolate all of the episodes where a mathematics teacher educator (MTE) who observed all lessons in the college algebra course intervened in the lesson to provide in-the-moment coaching to the PSTs. The second step was to have a trained rater on the project team use the Mathematical Quality of Instruction rubric to identify moments where PSTs made a mathematical error regardless of whether this resulted in an intervention by the MTE. This resulted in an initial collection of 5 possible episodes where PSTs had made mathematical errors. We then reviewed each of these instances to develop open codes to describe the error that had been made. In addition, we reviewed feedback that the MTE had provided to the P1, P2 and P3 lesson planning artifacts and isolated all instances (n=21 comments) where the MTE commented on mathematical content issues.

Two iterations of refinements to the coding categories resulted in four codes to describe the types of content-related errors PSTs were making in their planned or enacted instruction. Instances coded as *Content Error Correction* required PSTs to have made an explicit mathematical error that needed correction. For instance, one PST pair had written in their lesson plan that "negative exponents create fractions." The instructor was quick to point out, however, that “negative exponents invert fractions,” making sure the PSTs understand that non-whole numbers also can be taken to a negative exponent. Instances coded *Mathematical Precision* included feedback or interventions that reminded PSTs to be careful about the language they use or the instructor asked clarifying questions to clear up parts of the lesson plan that were not immediately clear mathematically. The code *Knowledge of Content and Students Suggestion* included instructor comments suggesting alternate phrasing or terms in order to avoid confusion for the students while also providing justification by connecting the comment to students’ prior knowledge or broader knowledge of the content as it is taught in schools. Lastly, *Typo/Other* included comments that corrected a simple typographical error or comments that were otherwise different from the rest.

In addition to assigning these codes, we accounted for the mathematical topic of the lesson, and whether the error during enactment was during content presentation or originated in response to a question from students.

Results

Table 1 shows the mathematical content addressed during the implementation interventions or in the lesson plan feedback. The most common mathematical areas where content errors occurred were in the areas of Functions and their Inverses (n=5), Composition of Functions (n=4), and Solving Exponential Equations (n=3). At first glance, one could see high error numbers as being a result of particular mathematical content being more difficult. It might also likely be a result of particular pairs finding difficulty in planning or teaching the content. The data suggest both of these conclusions are plausible; errors working with inverse functions spanned across three PST teams, whereas all four errors in composing functions happened with one particular team of PSTs.

Table 1

Mathematical content addressed by interventions during planning and enactment

Content	**Number of Instances**
Functions and their Inverses	5
Composition of Functions	4
Solving exponential equations	3
Properties of Logarithms	2
Simplifying logarithmic expressions	2
Exponential vs Logarithmic Functions	2
Place Value for large numbers	1
Transformations, parent functions.	1
Perfect Squares	1
Definition of Logarithmic Function	1
Slope of a Line	1
Interpreting Variable meaning	1
Even, Odd functions	1
Exponential Growth	1
Negative exponents	1

Kinds of Content-Related Errors in Planning

All seven teams of PSTs received mathematical content error feedback on their lesson plans. Three teams of PSTs received feedback on their round two pre-instruction lesson plan documents, while six teams received feedback on their round three documents. Of the 22 feedback items coded as errors, 10 were coded as Mathematical Precision, 6 were coded as Content Error Correction, 4 were coded as Knowledge of Content and Students Suggestion, and 2 were coded as Typo/Other. Select examples of each error can be seen in Table 2.

Table 2
Sampling of lesson plan errors and feedback given by Teaching Assistant

PST Lesson Plan Error	**Teaching Assistant Feedback**	**Code**
Make observations about how logarithmic and polynomial functions are different.	Exponential functions – there are no logarithmic functions included in the number talk.	Mathematical Precision
A student may incorrectly generalize from their classes on exponents, that "negative values are impossible.""	I'm not sure what you mean by this – they haven't learned that negative values are impossible as exponents… Or do you mean that 10 to a power can never give you a negative value?	Mathematical Precision
This [standard] is used when students recognize that the x values and y values are constant when looking for slope (linear)"	The change in x values and y values is constant, not the values themselves.	Content Error Correction
The inputs of the first function equals the outputs of the second function and hence are inverses.	And vice versa – without also looking at the outputs of the first function and the inputs of the second function, you don't have enough information to say they're inverses.	Content Error Correction
Properties of Even/Odd Functions: Symmetric over the y axis (even) or origin (odd)	I would be careful with how you describe this – if the students are only thinking of reflective symmetry and not rotational symmetry, this could be pretty confusing.	Knowledge of Content and Students Suggestion
Also, just before the explore activity, we plan on having a "bridge" activity to list what they think log problems are.	Should [problems] be "properties"?	Typo/Other

Kinds of Content-Related Errors During Instruction

Very few mathematical errors occurred during instruction (n=5), and no errors were repeats of those addressed during the lesson planning phase. The low number of errors and lack of repetitive errors indicates that receiving feedback during the lesson planning phase was successful in preventing many instructional errors. Of the five errors requiring intervention from the mathematics teacher educator (MTE) observing their instruction, four were coded Content Error Correction and one was coded Mathematical Precision. There were also two styles of interventions that occurred. In three of the error instances, teaching assistants made inquiring questions or comments to assist the PSTs in recognizing their error and worked with the PSTs to correct themselves and move on more naturally. In the other two instances, however, PSTs had to take a more direct intervention approach where the MTE took over instruction in order to avoid student confusion. In both instances, PSTs resumed instruction when the MTE finished the explanation, continuing their instruction as planned. Intervention sequences were brief, with the longest being only 3 minutes and 20 seconds (and that one sequence included two separate errors requiring intervention).

Discussion and Conclusion

Although our sample size is small, the results suggest the need for further inquiry into fundamental conceptions that secondary prospective teachers hold about the mathematics they will be teaching. Existing literature documenting the nature of secondary mathematics' PSTs content knowledge for teaching is sparse, with a few studies in areas such as geometry (Herbst & Kosko, 2012) and rational number (Depaepe et al., 2015), yet nearly all of the existing work has focused on capturing knowledge through written assessment measures rather than assessing knowledge as it is used in instruction. However, this study, along with work by Snider (2016), begins to unpack the nature of secondary mathematics' PSTs content knowledge for teaching as it is used in instruction.

The findings suggest at least two areas worthy of further inquiry. First, given the prominence of algebra in the secondary curriculum, it is important to acknowledge that participating PSTs needed further support in developing their understanding of topics such as invertible functions, composition of functions, and properties of exponential and logarithmic functions. The fact that these topics are difficult for secondary PSTs is not surprising as these are traditionally topics that pose difficulties for students in college algebra. However, our findings show that the additional coursework the secondary PSTs completed to prepare them for teaching mathematics did not resolve their misunderstandings or, for instance, add to their awareness of using mathematically precise terminology when discussing these topics in instruction.

Second, relatedly, our research raises the question of how best to develop PSTs content knowledge for secondary mathematics instruction. If, ultimately, strengthening PSTs content knowledge as used during instruction is the goal, then more attention should be paid to both researching knowledge as it is being used as well as strengthening knowledge within the context in which it is being used. For example, many of the interventions by teacher educators in this case involved issues of using mathematically precise terminology, because being precise contributes to clear communication with students and minimizes opportunities for confusion. Yet, it is no surprise that PSTs might not have received feedback about mathematical precision in their mathematics coursework if the work they produced resulted in a valid answer. The key obligations of mathematics teaching (Herbst & Chazan, 2012), such as managing the learning needs of a classroom of individuals, that may elevate particular aspects of content knowledge as especially important for teaching. The design and implementation of "content-focused" methods courses might be particularly promising for not only addressing the question of developing content knowledge for teaching as it is used in teaching but also serving as a productive site for collaborations between mathematics educators and mathematics teacher educators.

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