Student Reasoning about Span and Linear Independence:
A Comparative Analysis of Outcomes of Inquiry-Oriented Instruction

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In this report we examine the performance and reasoning of span and linear independence of 126 linear algebra students who learned through a particular inquiry-oriented (IO) instructional approach compared to 129 students who did not. Students who received IO instruction outperformed Non-IO students on questions focused on span, but not on questions focused on linear independence. Our open-ended coding additionally suggested that IO students' concept images of span and linear independence were more aligned with corresponding concept definitions than those of Non-IO students.

Keywords: inquiry-oriented instruction, student reasoning, span, linear independence, learning outcomes

A growing body of research documents improved student learning outcomes in undergraduate science, technology, engineering, and mathematics courses that use active approaches to learning (Freeman et al, 2014). However, there is limited work that documents differences in *how* students reason about particular disciplinary ideas under particular instructional approaches. In this paper, we analyze differences in performance and reasoning about span and linear independence of students whose instructors received instructional supports to teach linear algebra in an inquiry-oriented way from those who did not. In inquiry-oriented (IO) approaches to mathematics teaching, students inquire into mathematics by working on carefully designed sequences of open-ended problems, and instructors inquire into students' thinking and use their ideas to drive the development and formalization of mathematical ideas to align with language and notation more conventionally used among the broader mathematical community (Rasmussen & Kwon, 2007).

Throughout this report, we will refer to students who learned through an IO approach as IO students, and we will refer to students learned through other approaches as Non-IO students. Our analysis uses data from an assessment developed to assess student performance and reasoning around core concepts in linear algebra (Haider et al., 2016). This report will focus on students' responses to two multi-part questions that offer insights into students' understanding of span and linear independence. In this study we will try to analyze two research questions: (1) How did IO and Non-IO students reason differently about span? (2) How did IO and Non-IO students reason differently about linear independence?

Literature & Theoretical Framing

Many works that examined how students reasoned about span and linear independence discuss findings related to the categories of algebraic and geometric interpretations. Students tend to be more comfortable with algebraic than geometric approaches, and often do not use geometric intuition when solving problems about span and linear independence (Bogomolny, 2007; Aydin, 2014, Ertekin, Erhan, Solak, & Yazici, 2010, Stewart & Thomas 2010).

Students often think of linear dependence in a variety of algebraic ways: in terms of free variables, pivot positions, or rows of 0's in the reduced row-echelon form (RREF); they often think of linear independence as meaning there are no free variables, or that vectors are not multiples of each other (Bogomolny, 2007; Aydin, 2014). A common theme in this literature is that many students treat linear independence as a process; some think of it in terms of the row reduction procedure and some connect it to the homogeneous linear system $Ax = 0$.

Stewart & Thomas (2010) also found that students tended to rely on algebraic approaches when solving problems involving span. Bogomolny (2007) found that for some students geometric and algebraic representations seemed completely detached. This was seen in students' attempts to provide a geometric interpretation of the span of the set of column vectors of a matrix; instead of giving a geometric representation of the span of the columns of the matrix A, some students found a geometric representation of the solution set of the homogeneous system $Ax = 0$. By definition, span does not require linear independence, but by involving this concept students successfully interpreted span as a subspace of certain dimension (Wawro, Sweeney, and Rabin, 2011).

In this work, rather than focusing on distinctions between algebraic and geometric interpretations for analyzing student reasoning about span and linear independence, we draw on a helpful theoretical distinction made by Tall and Vinner (1981) which offers language for differentiating the way individuals *think* about particular mathematical ideas (concept image) from formal mathematical definitions for particular mathematical ideas that are more conventionally accepted by the broader mathematical community (concept definition).

Data Sources & Study Context

Data for this analysis is drawn from a broader study (NSF #1431595/1431641/1431393) of instructors who received a set of three instructional supports to teach linear algebra in inquiry-oriented ways. These instructional supports were: curricular support materials (consisting of task sequences, learning goals, descriptions of common student approaches to tasks, and implementation notes and suggestions), a 16-hour summer workshop, and facilitated online work groups that met for one hour per week during the semester instructors implemented the curricular support materials.

For this study, we have analyzed a total of 255 assessments where 126 assessments were collected from students in IO classes and 129 were from students in comparable Non-IO classes. The linear algebra assessment is a paper-pencil based test and was administered as a post-test in IO and Non-IO classes. All students were given up to 1 hour to complete the test. The assessment carries 9 questions, which are combinations of multiple-choice and open-ended items, and the focus of the assessment is to capture students' conceptual understanding of linear algebra concepts. The assessment was designed in way that a calculator was not required to answer any question on the test. In this study, we focused on an in-depth analysis of students' reasoning on the assessment questions related to span (question 1) and linear independence (question 3; see Figure 1).

Questions Q1a and Q1b offer insight into how students interpret the span of a set of vectors as a geometric object; Q1c and Q1d offer insight into how students identify when particular vectors are part of the span of a set of vectors. The multiple choices for these items provide systematic insights on these students' concept images of span, whereas their open-ended responses have the potential to provide insights into connections to the concept definition. Question 3b explicitly asks students to justify their response to whether a given set of vectors are linearly independent by connecting the result of a procedure (row reduction) – which we also think will offer insights into the links between students' concept image and the concept definition of linear independence.

Instructors using the IO approach used a 4-task sequence developed to support students' reinvention of the concepts of span and linear (in)dependence (Wawro, Rasmusen, Zandieh, Sweeney, & Larson 2012). In task 1, students have two modes of transportation whose movement is restricted to correspond with two particular vectors in R^2 to try to arrive at a particular given location. In task 2, students explore whether it is possible to "get anywhere" in the plane using the

1. Answer the following questions regarding the set of vectors $V = \left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$.

   a. Which of the following best describes the span of the set V?

   - ☐ i: A point
   - ☐ ii: Two points
   - ☐ iii: A line
   - ☐ iv: Two lines
   - ☐ v: A plane
   - ☐ vi: Two planes
   - ☐ vii: A 3-dimensional space

   b. Justify your response to part a.

c. Which of the following are in the span of $V = \left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$? (Check all that apply.)

- ☐ i: $\begin{bmatrix}1\\2\\0\end{bmatrix}$
- ☐ ii: $\begin{bmatrix}1\\2\end{bmatrix}$
- ☐ iii: $\begin{bmatrix}0\\-2\\-4\end{bmatrix}$
- ☐ iv: $\begin{bmatrix}1\\0\\0\end{bmatrix}$
- ☐ v: $3.1\begin{bmatrix}1\\2\\0\end{bmatrix} - \frac{4}{5}\begin{bmatrix}0\\1\\2\end{bmatrix}$
- ☑ vi: any vector in $\mathbb{R}^3$

d. Explain in general how you can determine if a given vector is in the span of some other set of vectors.

3. Consider the set of vectors $W = \left\{\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\-3\end{bmatrix}, \begin{bmatrix}6\\-5\\7\end{bmatrix}\right\}$. To determine whether the set is linearly independent or dependent, a student did the following correct row reduction:

$$\left[\begin{array}{cccc|c}1&4&3&6&0\\-2&-1&1&-5&0\\3&0&-3&7&0\end{array}\right] \sim \left[\begin{array}{cccc|c}1&0&-1&0&0\\0&1&1&0&0\\0&0&0&1&0\end{array}\right]$$

   a. The set of vectors W is (circle one): linearly independent / linearly dependent.

   b. Explain what it is about the row reduced matrix that tells you whether or not the set $W = \left\{\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\-3\end{bmatrix}, \begin{bmatrix}6\\-5\\7\end{bmatrix}\right\}$ is linearly independent or dependent.

FIGURE 1. Assessment items related to span and linear independence

same two vectors; after students work on this task, the instructor formalizes the definition of span of a set of vectors as the set of all possible linear combinations of the vectors in the set. In the third task, students are given three modes of transportation in R^3 and explore whether it is possible to take a non-trivial journey using those vectors that starts and ends at home. Sets of vectors that allow such non-trivial journeys are linearly dependent – an idea the instructor leverages following task 3 to formalize the definition that a set of vectors is linearly dependent when the corresponding homogeneous vector equation has a non-trivial solution. In the final task, students work to try to generate examples of sets 2 and 3 of vectors in R^2 and R^3 that are linearly dependent and independent; students form and justify generalizations based on this example-generating activity.

Methods of Analysis

To identify differences between IO and Non-IO student' performance and reasoning about span, we first look quantitatively at response patterns to multiple choice questions to Q1a and Q1c, and then look qualitatively at open ended responses to Q1b and Q1d to better understand the nature of student reasoning and differences between IO and Non-IO students. For linear independence, we did a similar quantitative and qualitative analysis to Q3a and Q3b. Quantitative comparisons of response patterns between IO and Non-IO students on multiple choice items were made using z-tests to see if there were statistically significant differences in the proportion of choices that IO and Non-IO students picked for every item. To qualitatively see how IO and Non-IO students reasoned, we engaged in open coding by first examining a subset of student responses to identify the variety of mathematically distinct ways students reasoned about each open-ended response question; we continued analyzing additional responses, refining categories as we did so, until our categories were saturated. This process led to 7 main categories of students' reasoning about Q1b, 2 categories about Q1d and 6 about Q3b (see Table 1). Items that did not fall into the categories described in Table 1 were labelled as "other" or marked if they were left blank. Student responses could be coded in multiple categories. During the coding we also paid attention to these reasonings if they align with the definitions or not and assign them as correct reasonings, otherwise they were incorrect reasoning (For example, students who reasoned in terms of linear independence did so correctly if they wrote something like ‘the two vectors are linearly independent (or not scalar multiples of each other or not parallel …) so they make a plane.’) We also use z-test to compare the proportion of codes assigned to the responses in both groups.

Table 1. Codes for Q1b, Q1d and Q3b and their descriptions

Questions	Code	Description
Q1b (Span)	Linear Independence	Response indicates that the two vectors are linearly independent or are not (scalar) multiples of each other.
	Linear Combination	Response refers to a linear combination of the two vectors (either directly in words, by giving the formula $xv_1 + yv_2 = w$, or stating something like ‘getting anywhere’)
	Different Directions	Response indicates that the two vectors point in different directions.
	Row Reduction	Student row reduces a matrix comprised of the given vectors.
	Dimensionality	Response makes explicit reference to the number of vectors, entries, or pivots; or claims that the two vectors are a basis
	Vector as Point/Line/Plane	Student identifies each vector individually as corresponding to either a point, line, or plane
	Geometric/ Graphical representation	Response includes a drawing showing a geometric representation as a response or part of it.
Q1d (Span)	Augmented Matrix/Row Reduction	Student row reduces the matrix comprised of the given vectors and concludes the vector is/is not in the span if the result is consistent/inconsistent or there is / is not a solution.
	Linear Combination	Same description as in Q1b.
Q3b (Lin. Ind.)	Compares RREF to Identity Matrix	Response indicates whether row reduction leads to identity matrix, especially comparing number of rows/columns
	Pivots	Response indicates if there are missing pivots in one or more columns/rows, if there is a pivot in every column/row, or explicitly references number of pivots
	Linear Combination	Explicitly or implicitly observes that one of the columns is a linear combination of other columns
	Solving Ax = 0	Response refers to solutions to the equation Ax = 0, e.g. non-trivial or infinitely many solutions
	# columns > # rows, or # vectors > $\dim(R^3)$:	Response indicates the number of columns or vectors is bigger than the number of rows or the dimension of R^3, or that the matrix M is not square
	Free Variable	Response explicitly indicates there is a free variable

Findings

Our quantitative analysis of the multiple-choice questions showed that IO students outperformed Non-IO students on span questions, but not on linear independence questions. Our open-ended coding additionally suggested that IO students' concept images of span and linear independence were more aligned with corresponding concept definitions than those of Non-IO students. Additional details about trends in student reasoning follow.

Differences in IO and Non-IO Student Performance and Reasoning about Span

When asked to identify which best describes the span of a given set of two (linearly independent) vectors in R^3 on Q1a, almost twice as many IO as Non-IO students correctly selected "A Plane" (see Table 2). This difference was statistically significant ($p < .001$). All other choices (which are incorrect answers to the given problem) were picked at higher rates by Non-IO students than IO students; in the case of choices Two Points, A Line, and Two Planes this difference was also statistically significant.

TABLE 2. Popularity of choices of Q1a Picked by IO and Non-IO Students

Choices	IO (n=126)	Percentage (IO)	Non-IO (n=129)	Percentage (Non-IO)	Significance* (z-test)
i. A point	1	.79	1	.77	p=.984
ii. Two points	0	00	5	3.9	**p=.026**
iii. A line	4	3.2	12	9.3	**p=.043**
iv. Two lines	6	4.8	8	6.2	p=.617
v. A plane	94	74.6	53	41.1	**p<.001**
vi. Two planes	5	4	17	13.2	**p=.009**
vii. A 3-D space	12	9.5	14	10.9	p=.726

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

When comparing the reasoning of IO and Non-IO students, we note two key trends. First, IO students were significantly more likely to reason about span in terms of linear independence, dimensionality, or row reduction than Non-IO students, and they employed these forms of reasoning *correctly* at much higher rates. Non-IO students on the other hand, were significantly more likely to interpret the span of a set of vectors by interpreting each vector individually as a geometric object. (This is consistent, for example, with significantly more Non-IO students selecting "Two points" and "Two planes" on Q1a.) Table 3 summarizes the coding of justifications students gave for their choices on Q1a; the p-values provided regard the comparison of the number of IO and non-IO students who used an approach (not the number using it correctly).

TABLE 3. Codes for IO and Non-IO Students' Approaches to Q1b

Codes	IO students |# used correctly (n=126)	Non-IO students |# used correctly(n=129)	Significance* (z-test)
Linear independence	53(42%) | 51(40%)	28(20%) | 27(21%)	**p<.001**
Linear Combination	22(17%) | 19(15%)	18(14%) | 16(12%)	p=.441

Different Directions	7(6%) | 7(6%)	3(2%) | 3(2%)	p=.183
Row Reduction	10(8%) | 6(5%)	0(0%) | 0(0%)	**p=.001**
Vector as Point/Line/Plane	21(17%) | 10(8%)	36(28%) | 6(5%)	**p=.032**
Geometric/Graphical	23(18%) | 15(12%)	17(13%) | 9(7%)	p=.267
Dimensionality	51(40%) | 42(33%)	32(25%) | 21(16%)	**p=.008**

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

When asked to identify whether or not particular vectors lie in the span of a set of vectors, IO students were significantly more likely to select choices that were a scalar multiple of one of the vectors in the set (iii) or explicitly expressed as a linear combination of vectors in the set (v), (see Table 4.) On the other hand, Non-IO students were significantly more likely to incorrectly select the choice that indicates any vector in R^3 is in the span of the given set of two vectors.

TABLE 4. Popularity of Choices of Q1c Picked by IO and Non-IO Students

Choices		IO (n=126)	Percentage (IO)	Non-IO (n=129)	Percentage (Non-IO)	Significance* (z-test)
i.	$[1,2,0]$	107	85%	110	85%	p=.936
ii.	$[1,2]$	19	15%	24	19%	p=.453
iii.	$[0,-2,-4]$	101	80%	78	60%	**p<.001**
iv.	$[1,0,0]$	13	10%	22	17%	p=.119
v.	$3.1[1,2,0]-\frac{4}{5}[0,1,2]$	90	71%	77	60%	**p=.049**
vi.	Any Vector in R^3	10	8%	23	18%	**p=.019**

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

Table 5. Codes for IO and Non-IO Students' Approaches to Q1d

Codes	IO Students | # used correctly (n=126)	Non-IO Students | # used correctly (n=129)	Significance (z-test)
Linear Combination	99(79%) | 93(74%)	97(75%) | 81(63%)	p=.522
Augmented Matrix (RR)	27(21%) | 12(9.5%)	10(8%) | 5(3.9%)	**p=.002**
Other	5(4%) | 0(0%)	18(14%) | 0(0%)	**p=.005**
Empty	5(4%)	6(5%)	p=.787

We noted above, Q1a and Q1b provide insight into students' geometric interpretations and justifications. Q1c and Q1d provide insight into how students interpret span in terms of individual elements, i.e. how students decide if individual vectors are in the span of a set of vectors, as opposed to describing the entire span of that same set of vectors as a geometric object. Looking across these two questions, we note one key interesting story: in Qc, IO students pick correct choices, (especially scalar multiple and linear combination of vectors in the set are in the span of the set) at higher rates, suggesting they have a better sense of how to identify vectors in the span than Non-IO students. In Q1d, we see IO and Non-IO students use linear combination reasoning at similar rates, though IO students did so correctly more than Non-IO students. Based on results from Q1c, IO students have a more robust concept image of span (e.g. they have a better sense of the variety of forms this can take; scalar multiples and linear combinations). See table 5.

Differences in IO and Non-IO student performance and reasoning about linear independence.

When asked whether a given set of 4 vectors in R^3 is linearly independent or dependent (and given the correct RREF of the augmented matrix comprised of those column vectors), there was no statistically significant difference in the portion of IO and Non-IO students who correctly said the set was linearly dependent (see Table 6). However, there were some differences in reasoning of IO and Non-IO students.

Table 6. Choices Selected by IO and Non-IO Students on Q3a

Choices	IO Students (n=126)	Percentage (IO)	Non-IO Students (n=129)	Percentage (Non-IO)	Significance* (z-test)
Linear Dependence	101	80%	97	75%	p=.343
Linear Independence	19	15%	31	24%	p=.072

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

When justifying their responses about whether the set in Q3a was linearly independent, IO students were more likely to reason by comparing the number of rows/columns in the RREF (in comparison to the identity matrix), or in terms of the solution to $Ax = 0$, and more IO students reasoned correctly using those approaches. This suggests for IO students, there may be better alignment between their concept image and concept definition.

Table 7. Codes for Various Students' Approaches to Q3b

Codes	IO Students (n=126)	Used Correctly (IO)	Non-IO Students (n=129)	Used Correctly (Non-IO)	Significance* (z-test)
Compare RREF to I	20 (16%)	19 (15%)	6 (5%)	6 (5%)	**p=.003**
Pivots	31 (25%)	21 (17%)	42 (33%)	31 (24%)	p=.159
Lin. Comb	25 (20%)	22 (17%)	28 (22%)	20 (16%)	p=.711
Solving $A\bar{x} = 0$	31 (25%)	26 (21%)	17 (13%)	10 (8%)	**p=.020**
#Col > #Rows OR #Vectors > $\dim(R^3)$	14 (11%)	14 (11%)	19 (15%)	18 (14%)	p=.390
Free Variable	32 (25%)	30 (24%)	31 (24%)	27 (21%)	p=.802

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

Discussion

We found IO students outperformed Non-IO students on span questions, exhibiting a wider range of appropriate concept images of span. While IO students did not outperform Non-IO students on the linear independence question, our data suggests IO students' interpretations were more explicitly linked to the concept definition. Future work will further explore this issue.

References

Aydin, S. (2014). Using example generation to explore students' understanding of the concepts of linear dependence/independence in linear algebra. *International Journal of Mathematical Education in Science and Technology 45*(6), 813-826.

Bogomolny, M. (2007). Raising students' understanding: Linear algebra. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education.* Vol. 2, pp. 65-72.

Ertekin, E., Solak, S. and Yazici, E.. (2010) The effects of formalism on teacher trainees' algebraic and geometric interpretation of the notions of linear dependency/independency. *International Journal of Mathematical Education in Science and Technology 41*(8), 1015-1035.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410-8415.

Haider, M., Bouhjar, K., Findley, K., Quea, R., Keegan, B., & Andrews-Larson, C. (2016). Using student reasoning to inform assessment development in linear algebra. In Tim Fukawa-Connelly, Nicole E. Infante, Megan Wawro, & Stacy Brown (Eds.), *19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 163-177). Pittsburgh, PA.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of research in science teaching*, *48*(8), 952-984.

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*, *26*(3), 189–194.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012) An Inquiry-Oriented Approach to Span and Linear Independence: The Case of the Magic Carpet Ride Sequence, PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, *22*(8), 577-599.

Wawro, M., Sweeney, G., and Rabin, J. (2011). Subspace in linear algebra: investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics*. *78*(1): 1-19.