A Possible Framework for Students' Proving in Introductory Topology

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Advanced mathematics courses require that students possess sophisticated proving techniques. Topology is one such course in which students' proving behaviors have not been extensively studied. In this paper, we propose that visual methods play an important role in undergraduates' discovery of the key idea of a proof, and we describe a potential framework for students' proving processes in a first course in undergraduate topology based on Carlson and Bloom's (2005) problem solving framework.

Keywords: topology, proof, visualization, representation, key idea

Background

Proof is of great importance in mathematics, but it is known to be a difficult concept for students (Dawkins, 2016; Harel & Sowder, 1998). Harel and Sowder (1998) define *proving* to be "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). Indeed, proving is a composite of two processes: "Ascertaining is the process an individual employs to remove her or his own doubts about the truth of an observation. Persuading is the process an individual employs to remove others' doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241).

When proving, a mathematician's primary goal is often the discovery of the *key idea* of the proof: "A key idea is an heuristic idea which one can map to a formal proof with appropriate sense of rigor. It links together the public and private domains, and in doing so gives a sense of *understanding* and *conviction*. Key ideas show *why* a particular claim is true" (Raman, 2003, p. 323). A heuristic idea is an informal idea, often represented by a picture, which gives the *individual* an understanding of why a conjecture is true, but which may not lead to a rigorous proof. Determination of the heuristic idea may be the primary goal of visualization: "The drawing of a diagram was not a goal in itself but a means to aid them in gaining more information for the problem situation. Mathematicians *anticipated* that a figure would provide them with specific information – the drawing of a diagram was not simply a vague step forward in the solution of the problem" (Stylianou, 2002, p. 310). However, the key idea is necessary for the construction of a formal proof, as the prover must convince not only herself, but she must provide an argument which will convince others as well.

A diagram constructed in search of a heuristic idea may be thought of as a type of example. Watson and Mason (2005) define an *example* as "anything from which a learner might generalize" (p. 3). Students often use specific and generic examples to help make sense of a definition or theorem. Such examples make up part of the student's *example space* for the given topic (Mason & Pimm, 1984; Watson & Mason, 2005). This example space serves as a starting point when encountering definitions to be used in other contexts. Examples, along with definitions, theorems, actions, and images associated with an idea, constitute the individual's *concept image* (Tall & Vinner, 1981).

Moore (1994) observed that students often use definitions to generate examples. These examples then help to develop their concept image, which informs the students' understanding of the original definition. The chosen examples transition from a *model of* the definition to a *model for* the more sophisticated knowledge necessary for proof construction (Cobb, Yackel, &

McClain, 2000). Moore identified the scheme "Images → Definitions → Usage" to describe a successful trajectory used by students in his study. The scheme "Images → Usage" often failed students in his study. When examples were used to guide students toward a deeper understanding of a definition, the *definition* became more useful during proof construction.

Building on existing frameworks for individuals' problem-solving processes, this study proposes a framework for students' proving processes. Through observations of the proving behaviors of Stacey, an undergraduate taking a first course in topology, we propose a framework for students' reasoning in proving. Our results show that students' approaches to proving and problem solving are similar to those of experts but exhibit some key differences.

Theoretical Perspective

We examined our data using the Multidimensional Problem-Solving Framework (MPS Framework; Carlson & Bloom, 2005). Research into problem solving has shown that mathematicians use visual and analytic methods in a cyclic process to help them solve problems (Carlson & Bloom, 2005; Stylianou, 2002; Zazkis, Dubinsky, & Dautermann, 1996). A precursor to the MPS Framework, the Visualization/Analysis Model (VA model) describes a process of alternation between visual and analytical strategies employed when solving problems (Stylianou, 2002; Zazkis, Dubinsky, & Dautermann, 1996). The MPS Framework elaborates on this idea, proposing a cycle of four phases through which expert mathematicians proceed when solving problems: Orienting, Planning, Executing, and Checking. The VA Model is encapsulated in the Orienting and Planning phases, during which the mathematician familiarizes herself with the problem, often by drawing a picture or creating a manipulative, and comes up with a strategy to solve the problem. A sub-cycle of *conjecture-imagine-evaluate* takes place during the Planning phase. The strategy is applied in the Executing Phase, and in the Checking phase, the mathematician looks back at her work and determines if she was successful in solving the problem or if she needs to try another approach.

Visualization plays a pivotal role in the Orienting and Planning phases. The construction of an appropriate diagram not only helps the problem solver to make sense of the problem scenario, but we argue that it may lead to the realization of the *key idea* (Raman, 2003) of the proof, allowing one to transition from the Orienting phase into the Planning phase. Our data suggest that students progress through the same four phases of the MPS Framework that were observed in expert mathematicians, but that students' ways of executing and checking are different from those of experts.

Methods

Four students (three undergraduate and one graduate) taking an introductory course in topology participated in at least one weekly, hour-long "Group Study Session" in which the students were asked to prove a true statement and to disprove a false one. The first author acted as the facilitator for all Group Study Sessions. One undergraduate student, Stacey (all names used in this study are pseudonyms), attended all sessions: the data presented here focus on Stacey's behaviors throughout the semester. The facilitator attended all class sessions (excluding exams); proof tasks were chosen based on material that had been covered recently in class. Group Study Sessions were video recorded. Students were encouraged to speak aloud as they worked and to work together with other students in the session. To maintain an authentic study atmosphere, students were permitted to use textbooks and notes as they wanted. As compensation, the facilitator offered extra office hours for participants to receive help with topology.

Using deductive thematic analysis (Braun and Clark, 2006), codes were applied to the data. Initial coding focused on identifying instances of students producing drawings, generating examples, and writing proofs. During this round of coding, it became evident that Stacey (as the only student to be present for all sessions) frequently used drawings to help her visualize definitions or to represent aspects of the problem scenario, and that these drawings seemed to influence her proving strategies. Based on the results of this first round of coding, a second round of coding identified instances of students constructing examples or drawings related to a definition, instances of students arriving at the key idea of a proof, and instances of students monitoring their work (either checking their own ideas or checking with the facilitator for logical consistency), as well as evidence of students' transitions through the phases of the MPS Framework. The patterns observed in these data resulted in the Topology Proving Framework proposed in this paper.

Data

The data presented here focus on Stacey's behaviors throughout the semester. Because this paper describes a framework for the construction of proofs of true statements, we describe the "prove" condition from two sessions; future work will focus on the "disprove" condition. Though Stacey produced drawings in Session 1 and Session 4 when prompted to do so by the facilitator, she did not spontaneously produce a drawing until Session 6. We present here two examples of Stacey's proving activities.

Session 6: Prove: ***A subset* A *of a topological space* $(X, \mathcal{T})$ *is said to be* <u>dense</u> *in* X *if* $\bar{A} = X$*. Prove that if for each open set* $O \in \mathcal{T}$ *we have* $A \cap O \neq \emptyset$*, then* A *is dense in* X*. (Note: the notation* $\bar{A}$ *indicates the topological closure of A in X.)***

For Session 6, Stacey was joined by Tom. The idea of a *dense subset* had not been discussed in class prior to this session, and Stacey had not previously encountered this idea. Tom had previously encountered this term in his introductory analysis course. After a brief reading of the problem, Stacey began by silently producing the drawing in Figure 1A.

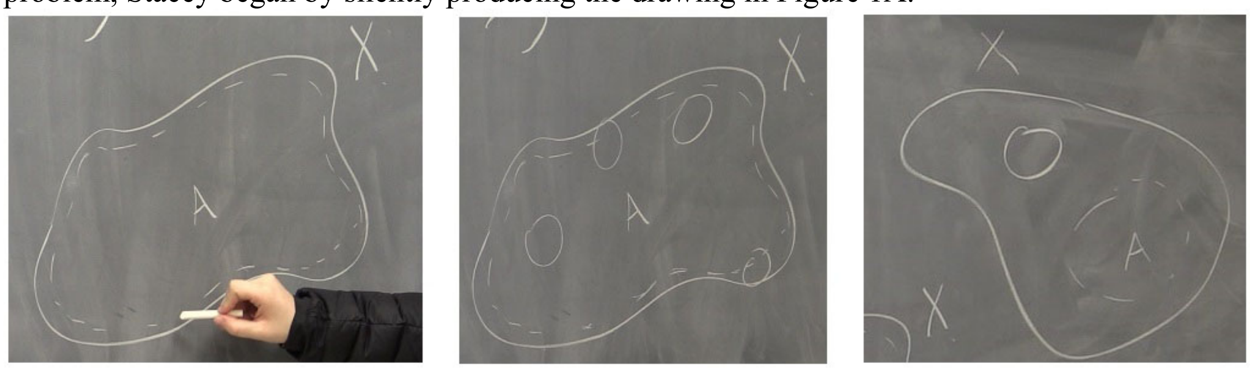

Figure 1A-1C: Stacey's drawings of a dense subset A of X. Figures 1A and 1B represent a dense subset; Figure 1C shows a subset A of X that is not dense.

After Stacey finished drawing, she explained:

> I can't really show it with a picture because I can't draw, like, a dashed line over a straight line, or like, a solid line, but we have X on the outside, and then we have the set A, which is represented by the dashed, which I wish I could get closer to this [pointing to the border of X], but I can't. So if we had the closure of A, then it would just be the same as that solid line [tracing the border of X with her hand]. So then if you take any open set

[drawing circles on her diagram, Figure 1B] anywhere, there has to be some kind of intersection with A. So if it wasn't, like if you take… if the intersection could be closed, er, could be, not closed, um, the empty set… [draws the diagram in Figure 1C] You've got X here… and A here, and you could have an open set here, and their intersection would be the empty set. [*code: recognizes key idea*] But then this closure wouldn't be equal to X. I get it conceptually I think, but I'm not sure how to prove it.

The preceding quote was coded as Stacey orienting herself to the problem. In the following excerpt, we see her transition into the Planning phase:

Stacey: We probably have to use the definition of closure in it… So we could say like… take x in… I don't know, either A or X, I'm not sure which one… and then a neighborhood of that point x…

Fac: Is there maybe a general strategy that you're thinking about? Or how are you thinking about approaching this problem?

Stacey: Um, I think contradiction, that's what's in my head right now.

Fac: If you had to outline your procedure – I know you don't have the whole thing fleshed out, but – how would your contradiction look? How would you set that up?

Tom: For the contradiction for this statement, it's gonna be "For each open set O, we have this *[points to* $A \cap O \neq \emptyset$*]*, but A is not dense in X. So the closure of A is not X." Right?

Following this exchange, several minutes were spent trying to determine whether the point *x* should be chosen from the set $\bar{A}$ or from *X*. Once it was agreed upon that *x* should be chosen such that it lies in *X* but not in $\bar{A}$, Stacey and Tom collaborated to write their proof. Stacey wrote "Let $x \in X$ and $x \notin \bar{A}$." Tom contributed, "So when you have this, when you have *x* is not in the closure of *A*, it means there is a neighborhood of *x* where it, intersect with *A*, will give the empty set," looking to the facilitator for confirmation of his reasoning. He followed this up by saying, "It doesn't seem right," but wrote this statement on the board, calling this neighborhood *N*. He then said this was the contradiction: "Now you have an open set that, intersect with *A*, gives you, uh, empty set." When the facilitator asked if *N* was an open set, Stacey concluded the proof by responding, "[The open set] is within the neighborhood… So there's *O*, subset of *N*, whose intersection with *A* is equal to the empty set." This resulted in a correct proof.

Session 8: Prove: Let $(X, \mathcal{T})$ be a topological space. A <u>separation</u> of X is a pair U, V of disjoint open subsets of X whose union is X. X is <u>connected</u> if no separation of X exists. If the sets C, D form a separation of X and if Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.

This was Stacey's first encounter with the idea of a *separation* of a topological space. Stacey was the only participant in Session 8. She began by drawing the diagram in Figure 2A to orient herself to the problem.

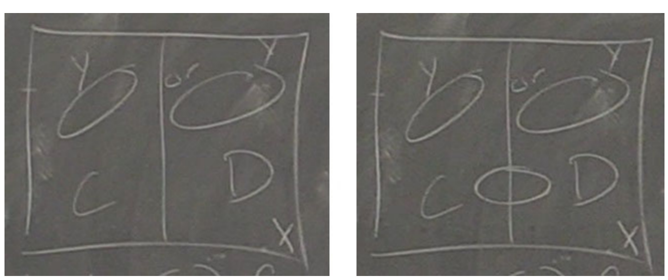

Figure 2A-2B: Stacey's drawings of a separation and a connected subspace Y.

She explained,

> If you have the *X*, the ambient space, and then you have the sets *C* and *D*, they form a separation, so that means that they're disjoint, so they don't have any of the same elements, and that their union is *X*, so that is satisfied for this. And then if *Y* is connected, which means it's not in these sets that are disjoint whose union is *Y*, it's just one cohesive set, then it has to be either in *C* or *D*, it can't be in both. Because if, if it was like that [draws the subset in Figure 2B], it would be disjoint. [*code: recognizes key idea*]

Stacey misspoke at the end of this explanation; throughout this session, she frequently said "disjoint" instead of "disconnected." In this last sentence, we observe Stacey's transition from Orienting to Planning.

She then began Executing her strategy, proceeding with her proof by way of contradiction. The facilitator provided guidance with logic and notation. Stacey frequently expressed correct ideas, such as the necessity to assume (for a contradiction) that some elements of *Y* lay in *C* and some elements lay in *D*. However, her initial notation read "Let $Y \subseteq C$ and $Y \subseteq D$." Because Stacey verbalized correct ideas, such as "If we do it like, by contradiction, and we say that there's intersection with both of them, and then we could show that *Y* can't be connected," we attribute errors like this to a lack of experience writing formal proofs, and specifically inexperience writing proofs in topology, rather than to a lack of understanding of the underlying ideas. When she changed her notation to a more appropriate statement, she checked with the facilitator to ensure that her new statement was accurate.

Stacey continued reasoning through the proof:

> Then you would say that… the points x and y are in disjoint spaces… From our assumption that C and D form a separation… So that would mean that… Y would have to be [disconnected] as well… Is there some kind of definition that says, like, a [disconnected] space that intersects all parts of another [disconnected] space is also [disconnected]? Is there something like that?

Stacey was talking about the fact that *Y* intersects both components *C* and *D* of *X*, which leads directly to the desired contradiction, as the sets $Y \cap C$ and $Y \cap D$ form a separation of *Y*, contradicting the connectedness of *Y*. As before, she appears to have the correct idea, but she lacks the experience to know exactly what she can do and how to formulate it correctly.

Discussions and Conclusions

The data presented here led to the creation of the Topology Proving Framework. It should be stressed that this is merely a *potential* framework; the small number of participants in this study makes it impossible to make generalizations with any reliability. This framework resembles the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005) in that it retains the idea of the four phases: Orienting, Planning, Executing, and Checking. Recall Stacey's behavior in Session 8: Stacey began investigating this conjecture by drawing a diagram to represent a separation, a clear sign of orientation to the problem. She then put forth the idea of proof by contradiction: *What if Y has intersection with both C and D?* "If we do it like, by contradiction, and we say that there is intersection with both of them, and then we could show that Y can't be connected." Here, Stacey has moved into the Planning phase and shows evidence of the sub-cycle of *conjecture-imagine-evaluate*.

Stacey's time in the Orienting phase often took a particular form. Beginning in Session 6 when she first began to produce drawings without prompting, her drawings frequently began as a visual representation of a key definition in the conjecture, occasionally becoming a representation of the entire problem scenario. As was reported by Stylianou (2002), this appears to have been a directed effort: the drawing seemed to stimulate Stacey's entry into the *conjecture-imagine-evaluate* cycle in the Planning phase, as it facilitated her ability to consider *What if?* questions. Furthermore, it is at this point that Stacey most frequently recognized the key idea (Raman, 2003) of the proof. For instance, in Session 6, Stacey drew two diagrams: one to represent a dense subset and one to represent a subset that is not dense. This seemed to motivate her to choose the strategy of proof by contradiction, and to recognize that if A is not a dense subset of X, then there must be some open subset O of X such that $A \cap O = \emptyset$. Ideas like this one do not always come fully-formed, as we saw in this example where Stacey seemed to have only a vague notion that contradiction *should* work. There was no guarantee that Stacey would necessarily know *how* to implement the key idea right away, as in this instance, in which Stacey wanted to begin her proof by choosing a point *x* which lay in either the set *A* or in the set *X*, but the determination of which set would be more productive seemed to require significant effort.

An interesting twist on the MPS Framework (Carlson & Bloom, 2005) as applied to Stacey's behavior arose when she entered the Executing and Checking phases. Carlson and Bloom's data show that experienced mathematicians proceed through the Planning, Executing, and Checking phases in a cyclic fashion until the mathematician is satisfied with her solution. Stacey, on the other hand, typically established a plan and then *alternated* between Executing and Checking activities. Furthermore, the experienced mathematicians Carlson and Bloom interviewed relied on their own *internal* resources to check their work. As a relative newcomer – not just to topology, but to proof writing in general – Stacey frequently checked with the *facilitator* to confirm notation, phrasing, and logical consistency, as seen in the following exchange from Session 5, in which part of the "Prove" condition asked Stacey to prove that the empty set and the set *X* are both closed in the topological space $(X, \mathcal{T})$:

Stacey: X is in $\mathcal{T}$, and then X is open, by definition.

Fac: Correct.

Stacey: And then the complement of X is the null set, and that's… closed.

Fac: Because…?

Stacey: Because… um… I mean, the null set is just like one, it's one element…

Fac: What reasoning did you apply to get there? X is in $\mathcal{T}$ …

Stacey: X is in T, so X is open. Well… So is the null set also in $\mathcal{T}$?

Fac: By definition, right?

Stacey: So that would also be open. So it's an open set… and the complement is the null set… And then the null set's also open, so then… it's a closed set?

Fac: Yeah.

Stacey: Same thing the other way around? So the null set can be open or closed, depending on the situation?

Fac: Well, not open or closed, but it's open and closed, simultaneously.

Stacey: But it's a different kind of open and closed than this, right? [points to the interval [0,1)]

Through her verbalizations, Stacey demonstrated the ability to self-monitor; external validation from the facilitator was not always necessary. However, this sort of external checking was common for Stacey, and it typically happened in conjunction with the execution of her proof construction, as part of an ongoing process of *Execute-Check-Execute-Check* which continued until the conclusion of her proof. We observed this kind of behavior with Tom as well in Session 6, as he unpacked what it meant for *x* to lie outside the closure of the set *A* while looking to the facilitator for confirmation of his reasoning.

The combination of these observations led to the development of the following Topology Proving Framework (TPF).

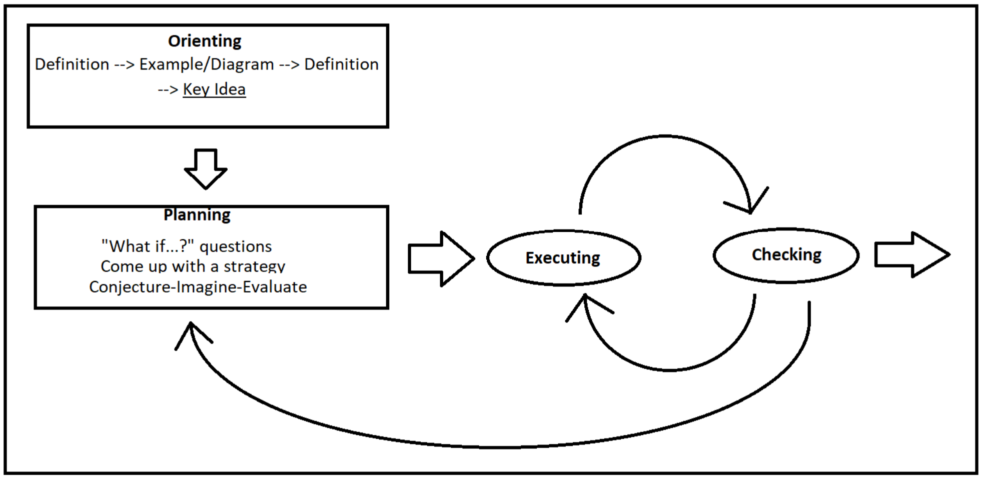

Figure 3:The Topology Proving Framework.

In keeping with the MPS Framework, the TPF begins with the student orienting herself to the problem. Most often, this took the form of the student converting a definition into a diagram or coming up with examples which gave a better understanding of the definition. This led to the realization of the key idea of the proof, which allowed the student to transition into the Planning phase.

With a visual representation of the key definition, the student was better equipped to ask *What if?* questions and to develop a plan, such as using proof by contradiction or direct proof. The recognition of the key idea gave the student a sort of "target," a sub-goal which, if proved, would result in the completion of the required proof. With a plan in mind, the student then began attempting to execute this plan. The execution was not always smooth and sometimes required some intense thought or trial-and-error. Throughout the execution of the plan, the student performed monitoring activities to ensure that she was still making progress toward her goal. These activities sometimes took the form of internal checks within herself, and other times they occurred as dialogue with the facilitator. Such external validation is not uncommon for students learning to prove or learning to prove in a specific content domain (Harel & Sowder, 1998). The alternation of execution steps and checking steps continued until a check resulted in the recognition of an error (which may reset the process back to the Planning phase) or in the student's perception that the proof was complete. The results of this study indicate that leveraging a key definition through visualization may be critical to success in identifying the key idea and producing a satisfactory proof in topology. Our future work will examine how this cycle is similar and different when tackling statements that require disproof.

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