# Mathematical Knowledge for Tutoring Undergraduate Mathematics

Linda Burks
Santa Clara University

Carolyn James
University of Portland

*Undergraduate math tutoring is an important venue for student learning, yet little empirical work has been done to study tutoring interactions and few theories specifically address tutoring interactions. Drawing upon literature from problem solving, peer learning, and mathematics teaching, this report proposes a schema for Mathematical Knowledge for Tutors (MKTu). The proposed framework expands Ball's (2008) Mathematical Knowledge for Teaching by adding dimensions of affect and self-regulation. This additional depth reflects the individualism, immediacy, and interactivity which are unique to the tutoring setting where problem solving and mentoring take place between an advanced undergraduate tutor and an undergraduate student.*

*Keywords:* Undergraduate Mathematics Tutoring, Affect, Self-Regulation, Problem Solving

Tutoring has long been recognized as an excellent form of education. The results of Mathematics Association of America's national study of college calculus indicate that 97% of the 105 American institutions surveyed had a tutoring center for students to receive help for Calculus, and 89% of the institutions offered tutoring by undergraduate students (Bressoud, Mesa, Rasmussen, 2015). While undergraduate mathematics peer tutoring is common, the research community is just beginning to focus on this critical out-of-classroom learning context. Several quantitative analyses indicate tutoring is associated with higher final grades (Byerly & Rickard, 2018; Rickard & Mills, 2018; Xu, Hartman, Uribe & Menke, 2014). To understand why tutoring is effective must include a better understanding of the mathematical knowledge necessary for effective tutoring. In this paper, we consider how Ball's (2008) Mathematical Knowledge for Teaching (MKT) framework might be adapted to apply to undergraduate mathematics tutors. Like Ball we question, "What do [undergraduate tutors] need to know and be able to do in order to [tutor] effectively. Or, what does effective [undergraduate tutoring] require in terms of content understanding?" (Ball et. al., 2008, p.394) In our contribution, we describe how the components of Ball's MKT construct translate to tutor knowledge, and we add dimensions to reflect the knowledge specific to an undergraduate tutoring context.

We focus on the knowledge of undergraduate tutors because it is ubiquitous, but also unique. Undergraduate math tutoring in this paper refers to peer tutoring in which a more experienced (typically upper-class) undergraduate student provides tutoring to another undergraduate math student. Peer tutors' knowledge differs from both mathematics instructors and fellow classmates; their experience bridges the gap between those with substantial subject matter knowledge and those of peer learners.

## Tutoring is not Teaching

Mathematical Knowledge for Teaching (Ball et. al. 2008) is well established for elementary students and has been extended to secondary and undergraduate teaching (Speer, 2015; Hauk, 2014). Still, a different type of mathematical knowledge is needed for tutoring undergraduate mathematics. As Mills, director of NSF funded mathematics resource center workshops, regularly reminds the tutor research community, “The application of teaching theories to tutoring likely results in a deficit model” (2018, unpublished manuscript). Unlike teaching, tutoring is not a profession; undergraduate tutors typically work for 1-3 years. Teachers typically have extensive pedagogical training; tutors may have experience teaching, but enter the job with no formal teacher training. Undergraduate math tutors have different math backgrounds from each other as well as different math backgrounds from trained teachers. Lastly, tutors’ understanding of mathematics curriculum commonly differs from instructors. While undergraduate math instructors have a good sense for the math content which they teach, undergraduate math tutors have a unique sense of how their math courses connect to courses in their particular major.

In addition, the tutoring context is substantially different from the classroom context: the instructional goals of each context may differ, and the knowledge required to meet those goals also differs. In the classroom, an instructor is responsible for teaching new material to many learners at once. In the tutoring context, the learner has some previous familiarity with the content, and the tutoring interaction typically takes place in an individualized setting with a focus on solving problems. The individualized tutoring context also allows for immediate, individualized feedback, while a classroom context typically cannot allow immediate feedback to all learners. A key role of the tutor is to help the student become an independent learner; thus prioritizing the development of self-learning skills over the mastery of content (Marx, Wolf, Howard, 2016). Although self-learning is valued in the classroom, most formative assessments in mathematics prioritize proficiency with content (Burn & Mesa, 2015). The power dynamic between a tutor and student is also likely different than that between a student and instructor. While undergraduate peer tutors are not peers in the strictest sense: they typically have slightly more math background than the students they are tutoring, and they relate more closely than an instructor does to a student.

Given these differences between tutor and teacher, an extended model for the Mathematical Knowledge for Tutoring Undergraduate Mathematics (MKTu) is needed and is relevant to the RUME community. To advance a research agenda aimed at describing and improving tutoring practices and tutor training, a theoretical model is needed to describe the mathematical knowledge necessary for undergraduate tutoring. The schema for MKTu proposed here builds on Ball’s MKT and is based on tutoring observations (McDonald and Mills, 2018; James and Burks, 2018), tutoring literature, problem solving theory, and peer learning methodology. The proposed framework is a theoretical contribution grounded in existing literature. This framework will require ongoing refinement based on empirical studies, and will help guide the focus of qualitative analysis of tutor actions and interactions.

## Mathematical Knowledge for Teaching

Ball describes and illustrates this theory of Mathematical Knowledge for Teachers (MKT) using the well-known “egg” diagram seen in Figure 1. Following Shuman’s (1986) analysis, Ball divides MKT into Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK is then further divided into Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). Common Content Knowledge is math knowledge which teachers use in ways similar to the way it is used in other occupations. Horizon Content Knowledge is cognizance of how mathematical concepts are related across the curriculum. Specialized Content Knowledge (SCK) is content knowledge specifically used by teachers. Similarly, PCK is subdivided into Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC). Knowledge of Content and Students (KCS) includes recognition of student misconceptions and reasoning how to build new understanding on student's current thinking. Knowledge of Content and Teaching (KCT) is knowledge of teaching moves. Knowledge of Content and Curriculum (KCC) indicates awareness of when a particular topic is first covered and then revisited within the elementary curriculum. While Ball's focus has been on elementary math teaching, others have applied it to secondary and tertiary math teaching (Speer, 2015; Hauk, 2014).

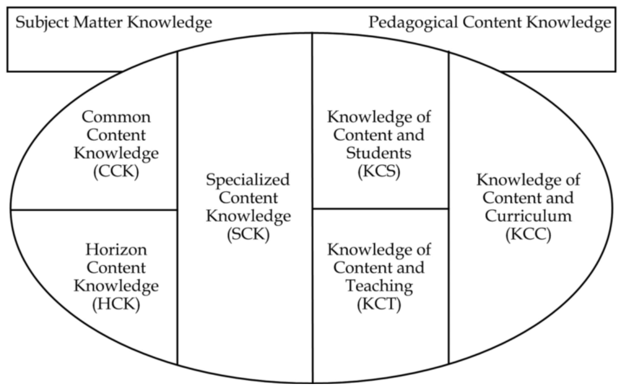


*Figure 1. MKT from Ball, Thames, Phelps (2008, p.403)*

## Mathematical Knowledge for Tutoring

Based on supporting literature, Ball's original MKT model is modified to reflect the mathematical knowledge uniquely representative of tutors. Our theoretical proposal extends Ball's egg to include affective and self-regulatory components. Ball's planar egg becomes a cross section of a 3-dimensional ellipsoid that forms the MKTu framework as seen in Figure 2. The 2D cross section looks very similar to the Ball framework; however, a lower supporting affective arc is laid underneath and a guiding overarching edge of self-regulation is added above. The planar cross section is discussed first, followed by the lower and upper arcs of affect and self-regulation. The three divisions of SMK for teaching remain in the planar cross section of SMK for tutoring: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK).

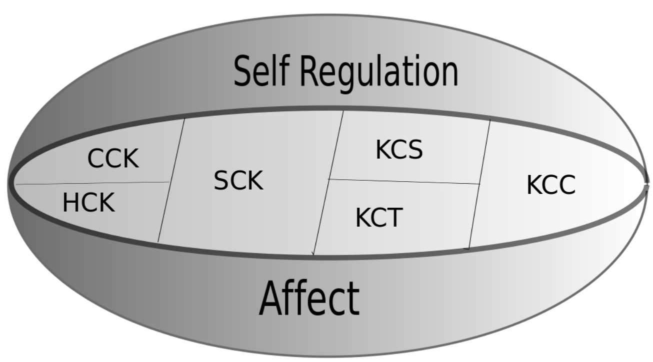


*Figure 2. Mathematical Knowledge for Tutors (MKTu)*

## Subject Matter Knowledge for Tutoring

Within SMK, a tutor's Common Content Knowledge tends to focus primarily on *knowing how* (Mason, 1999), which includes identifying the approach needed to solve the problem and subsequently carrying out the appropriate computations correctly. In contrast, a classroom instructor must draw many knowledge types -- knowing *how*, *why*, and *that* (Mason, 1999)-- while explaining concepts, carrying out procedures, and solving problems in the course of classroom instruction.

In addition, a tutor's CCK may differ significantly in scope compared to a teacher. Undergraduate tutors cannot be expected to have the depth and breadth of understanding of an undergraduate curriculum common to instructors; however, Common Content Knowledge may not be as critical to the tutoring experience. Tutors do not introduce new material; instead their primary role should be to encourage students to make use of their own resources (such as the textbook and class-notes), and guide students through the process of articulating their own self-explanations (Chi, 2008). Since the role of the tutor is different from a teacher, the type of content knowledge required for effective tutoring differs as well.

Common Content Knowledge and Specialized Content Knowledge for tutors is commonly characterized by the dominate role of problem-solving within the undergraduate tutor context. Students may be aware of concepts taught in class and may also have mastered topics from previous courses, but it is the tutor who helps students refine problem solving skills and build connections between prior knowledge and current knowledge. Whereas HCK for teachers considers how a current math topic fits in context with *math* curricula from prior and future years, HCK for tutors connects current math work with aspects of a specific *major* curriculum. For example, a tutor engineering major has knowledge of how integration theory is used in junior-level engineering courses in a way that a math instructor might not.

## Pedagogical Content Knowledge for Tutoring

Within Pedagogical Content Knowledge for tutors (PCKtu), Knowledge of Content and Students (KCS) for tutoring includes identifying and understanding student mathematical contributions to progress the mathematical agenda. This type of knowledge is very similar to the KCS for teaching described by Johnson (2012), which focused on the listening needed for instructors who enact inquiry-oriented mathematics. Like inquiry-oriented instructors, tutors also draw on SCK to make mathematical sense of students' contributions. However, unlike instructors who leverage student contributions toward a specific mathematical goal for the whole class, the tutor is interested in redirecting the student's ideas in a way that allows the student to engage in self-reflection to solve a specific mathematical task.

Additional components of KCS for tutors include *cognitive conflict*, *scaffolding*, and *error management* (Topping, 1999). In drop-in tutoring sessions, tutors must know how to effectively balance cognitive conflict; it is beneficial for a student to be productively confused, but harmful for a student to be hopelessly confused (Graesser, 2011). In addition, tutors must use error management to identify student conceptions and tailoring questions to lead a student to reflect upon and reform those conceptions when needed (Topping, 1999). Knowledge of effective scaffolding is also an important component of a tutor's KCS (Chi, 1996); it differs from classroom scaffolding primarily because of the individualization required to adapt to a specific student's problem-solving approach, rather than a whole-class scaffold.

Communication, organization, and engagement are all critical components within a tutor's Knowledge of Content and Teaching. Topping (1999) identifies communication as critical to effective tutoring, which includes listening, explaining, and questioning. This type of communication differs significantly from the communication found in a classroom: while an instructor must facilitate dialogue among many voices (Gay, 2002), the tutor must manage one-on-one interaction. *Organization and engagement* (Topping, 1999) captures the importance of active learning in the tutoring session. Topping includes goal setting, planning, time on task, the opportunity for individualization of learning, and immediacy of feedback within organization and engagement. In the mathematics tutoring setting, tutor and student goals for the tutoring session and selection of appropriate problems are included in the process of *organization and engagement*. In the context of problem solving, *organization and engagement* also includes the tutor-student dialogue taking place at each phase of problem solving: orienting, planning, executing, checking, monitoring.

Since tutors are more of a mentor than a peer, KCC comes from the tutors' extended experiences of courses and university culture. Tutor knowledge will arise from personal experience with the curriculum. Tutors have first-hand experience with the curriculum in their major and department; whereas math instructors have a knowledge of the content in context of the mathematics curricula.

**Affective Knowledge for Tutoring**

In this theoretical framework, we utilize Philipp's (2017) definition for *affect*: affect is "a disposition of tendency of an emotion or feeling attached to an idea or object. Affect is comprised of emotions, attitudes, and belief" (p. 259). Understanding how affect relates to mathematical learning is indeed an important component of mathematical knowledge for teaching as well as tutoring. However, affect plays a different, more prominent and more fundamental role, in MKTu. In particular, motivating students and helping them to cope with frustration are two key components of tutoring (Topping, 1999). This role of managing student affect, unique to tutors, is seen in motivation and emotions. Affect is so critical to the tutoring context that it is displayed as an arc underlying both Subject Matter Knowledge and Pedagogical Content Knowledge for Tutors in Figure 2.

Experienced problem solvers effectively work through an intense emotional cycle as they simultaneously work through a cognitive problem-solving cycle (McLeod, 1989). Awareness that affective elements are part of the mathematical problem-solving process is part of CCK for tutors. Confident students begin to work on a problem with enthusiasm. If they get stuck carrying out a plan, they may get tense and grow more frustrated with each attempt that leads nowhere. If they reach a solution, they experience the satisfaction, and possibly even delight, of an 'Aha' experience (Schoenfeld, 1992; Carlson, 2005). In the less ideal situation where students do not reach a solution, their frustration may turn to anger. If simmering, this anger may interfere with the effectiveness of a tutoring session. Knowing how to support students in their emotional responses is part of both KCS and KCT.

Motivation is another meaningful part of the affect arc supporting MKTu. Tutors need to have knowledge of what elements of the mathematics are motivating for students, which is another type of KCS. Whereas a teacher provides motivation in the enacted problem-solving process found in classroom, the individualized context of tutoring means tutors have the opportunity to uniquely motivate a particular student (Lepper and Woolverton, 2002).

Since understanding and practicing mathematics can raise heightened emotions (Beilock & Maloney, 2015), a math tutor may need to handle intense feelings from a student. To do so adequately, a trusting relationship between tutor and student is essential. A competent tutor models enthusiasm and confidence, which the student notices, either directly or indirectly. Tutors help students move from anxiety and fear to perseverance, persistence, and resilience; this is part of PCK for tutors.

Math anxiety is a different than other anxieties; it is uniquely related to the discipline of mathematics (Dowker, Sarkar, Looi, 2016). Math anxiety bridges both the cognitive and affective domains. Tutors need to manage the relationship between what the student needs to motivate them and what the student needs to develop mathematical understanding (Lepper & Woolverton, 2002). In doing so, the tutor coordinates mathematical subject knowledge with pedagogical content knowledge.

In MKTu, the affect arc undergirds SMK as well as PCK. Evidence of affect is seen in almost every tutoring session (James and Burks, 2018; Graesser, 2011). In fact, tutors with no training can be effective (Leary et. al., 2013). This observation suggests that the focus, persistence, and affirmation, each of which a tutor naturally gives a student, are key elements of student success. And so, affect is represented as a supporting foundation of Mathematical Knowledge for Tutors.

**Self-Regulatory Knowledge for Tutoring**

*Self-regulation*, which includes metacognition and skills for self-control and decision-making, overlays both Subject Matter Knowledge and Pedagogical Content Knowledge for Tutors. Metacognition, which is the ability to think about one's thinking, is a particularly important while problem-solving (Schoenfeld, 1992). Effective problem solvers spend time understanding the problem, designing a plan to solve the problem, carrying out the plan, and reflecting back. As they progress through each problem-solving phase, efficient problem solvers are aware of their position in the problem-solving process and cycle back to a previous phase when needed. Once a solution is reached, the problem solver looks back at the solution, checks the work, reflects on its validity and makes connections to other work.

This knowledge is not unique to tutoring; however, because problem solving forms the basis of most tutoring interactions metacognition is particularly important for tutoring. Tutors not only need to have metacognition about their own problem solving (a type of Common Content Knowledge), they need to understand which components of the metacognitive process are challenging when problem solving (SCK). In addition, they must evaluate where a student is in their metacognitive process (KCS) and know ways to move the student to the next step (KCT). The individual and immediate nature of tutoring (Lepper & Woolverton, 2002) makes this type of knowledge of metacognition for tutors distinct from the knowledge typically found in teaching.

Because problem solving permeates SMK for tutors, the rich discussions of the importance of metacognition in problem solving, peer learning, and tutoring seem relevant; however, little evidence of metacognitive moves are observed in tutoring sessions (Graesser, 2011). Topping (1996) specifies metacognition as key component of peer learning. Graesser (2011) describes the metacognition of the tutor with respect to teaching the student; however, more work is needed to help tutors share the metacognitive aspects of problem solving with their students.

Metacognition is one component of self-regulation. Other components such as teaching students to be more self-regulated with respect to study skills indicate that self-regulation is also a critical covering of PCK. Self-regulatory study skills help students acquire knowledge, connect knowledge, and apply knowledge. Within the context of mathematics, self-regulation may include general study skills such as setting goals, planning to reach those goals, and assessing whether those goals have been obtained. The elements of individualization, immediacy, and interactivity, distinctive characteristics of tutoring, suggest that there exist techniques of self-regulation which are unique to tutoring (Lepper and Woolverton, 2002). In addition, peer tutors have a different power dynamic with the students they tutor compared to a teacher, and they may have unique, less evaluative ways to relate to students regarding self-regulation and study skills.

The relevance of individualization, immediacy, and interactivity in the tutoring session leads to a greater distinction between teaching and tutoring. An effective tutor makes a crucial connection between a student's cognitive model and motivational model. This connection may be congruent, independent, or conflicting (Lepper and Woolverton, 2002). If cognitive and motivational diagnoses are congruent and lead to the same approach, the situation is ideal. If cognitive and motivational diagnoses are independent, an approach used to address either cognitive or motivational needs will not affect the other. When cognitive and motivational diagnoses lead to conflicting approaches, the tutor needs to discern which approach is most appropriate at a given time. This astute decision-making process, which follows the complex assessment of cognitive and motivation needs of the students, is unique to tutors and validates the placement of self-regulation as an awning overlaying Mathematical Knowledge for Tutors.

## Conclusions

This report proposes an initial schema for Mathematical Knowledge for Tutoring of undergraduate mathematics; the schema is based on literature in problem solving, peer learning, and mathematics teaching and tutoring. Ball's (2008) model of Mathematical Knowledge for Teachers is deepened to include the important role of affect in MKTu and raised to highlight the particular role of self-regulation in MKTu. It is important to note that many of the types of knowledge proposed in this framework are already a part of the responsive, personalized teaching that takes place during one-on-one discussions during classroom interactions. However, the tutoring context is necessarily unique due to the context, goals of the interaction, and the breadth of tutor experience. Imposing a model of teaching knowledge onto tutors results in a deficit evaluation: in contrast, this model highlights the critical types of knowledge necessary for tutors while expanding the framework to capture types of knowledge outside of the typical role of a teacher.

The adapted egg raises research questions and lays the basis for formal observations of undergraduate mathematics tutoring. As findings from research studies of tutoring interactions emerge, the MKTu egg will evolve into a more complex theoretical framework integrating subject matter knowledge, pedagogical content knowledge, self-regulation, and affect. The modified framework will serve as an incubator of new research questions and further studies.

Ultimately, our goal is to leverage future research built on this framework toward the development of training materials for undergraduate tutors. These materials will be implemented, assessed, and tested. Updates to the MKTu theory will in turn generate new sets of observations and research studies. The momentum of this continuing cycle will propel our work in identifying, understanding, and implementing effective practices as well as developing and testing training materials for undergraduate math tutoring.

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