Instructors' and Students' Images of Isomorphism and Homomorphism

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This study uses thematic analysis to examine the conceptual metaphors used by two abstract algebra teachers to describe the concepts of isomorphism and homomorphism, both in interviews outside instruction and during class. These metaphors are compared to the metaphors used by their students to describe these concepts. While the two instructors utilized similar metaphors for isomorphism, they did not share metaphors for homomorphism. Further, when looking from interviews to instruction, there was again more alignment with isomorphism than with homomorphism, with metaphors used to discuss homomorphism during the interviews being less present during instruction than those used to discuss isomorphism. The students in these two classes appeared to incorporate the instructors' metaphors to varying degrees.

Keywords: Isomorphism; Homomorphism; Abstract Algebra; Conceptual Metaphors

Experts have identified isomorphism and homomorphism as two of the most central topics to abstract algebra (Melhuish, 2015). Although some research has been done on how students approach isomorphism (e.g. Larsen, Johnson, & Bartlo, 2013), research explicitly on students' understanding of homomorphism, or on instructors' understanding of and instruction on isomorphism or homomorphism, has been scarce. Thus, the purpose of this study is to examine teachers' and students' understanding of isomorphism and homomorphism through their use of conceptual metaphors and to examine how teachers' and students' metaphor usage is similar and different. Specifically I sought to answer the following research questions: (1) What conceptual metaphors do the teachers use to describe isomorphisms and homomorphisms and what relationship exists between these metaphors and the mathematical content in instruction? (2) What is the relationship between the mathematical content in instruction and conceptual metaphors the students use to describe isomorphisms and homomorphisms?

Related Literature and Theoretical Perspective

An isomorphism between groups is defined as follows: "The map $\phi: G \to H$ is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if ϕ is a homomorphism, and ϕ is a bijection" (Dummit & Foote, 2004, p. 40). A homomorphism between groups is defined as follows: "Let $(G,\star)$ and $(H,\boxdot)$ be groups. A map $\phi: G \to H$ such that $\phi(x \star y) = \phi(x) \boxdot \phi(y)$ for all $x, y \in G$ is called a homomorphism" (Dummit & Foote, 2004, p. 39). An isomorphism can be thought of as a function that preserves the structure of a group in another group of the same cardinality; a homomorphism also preserves the structure, but can be formed between groups of different cardinalities. Two groups may or may not be isomorphic, but there is always at least one homomorphism between groups: the trivial homomorphism, by which every element of G is mapped to the identity in H. Quotient groups link isomorphism and homomorphism through a theorem known by many names, including the Fundamental Homomorphism Theorem (FHT): "If $\phi: G \to H$ is a homomorphism of groups, then $\ker(\phi) \trianglelefteq G$ and $G/\ker(\phi) \cong \phi(G)$" (Dummit & Foote, 2004, p. 97).

A theoretical lens for analyzing mappings in general is the conceptual metaphor construct (e.g. Lakoff & Núñez, 2000). "Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another" (Lakoff & Núñez, 2000, p. 6). Conceptual metaphors have been used to examine students' reasoning about many topics including linear transformations and functions more broadly (Zandieh, Ellis, & Rasmussen, 2016). Zandieh and colleagues examined the properties and metaphorical expressions students used within five metaphorical clusters: Input/Output, Traveling, Morphing, Mapping, and Machine. While these clusters informed background knowledge, every effort was made to ascertain whether or not these clusters were appropriate for the specific concepts of isomorphism and homomorphism,. As isomorphisms and homomorphisms are particular types of functions, these metaphors offer a starting place for this investigation. However, in addition to the functional aspect of these concepts, there are also structural properties (e.g., groups can be *isomorphic*). Thus, considering the literature on how students reason about function is necessary but not sufficient.

Previous studies have examined isomorphism in problem-solving, proof, and teaching contexts. Early studies mostly provided students with two Cayley tables or stated two groups and asked if they were isomorphic or how they could tell they were isomorphic. Dubinsky, Dautermann, Leron, and Zazkis (1994) found that when students considered isomorphisms between groups, they considered the cardinality of each group, but not whether the homomorphism property was satisfied. Leron, Hazzan, and Zazkis (1995) noted students' tendency to check the cardinality of a group as well as a general utilization of "sameness" as a stand-in for isomorphism, terming this "naïve isomorphism." In related studies, Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to prove theorems related to isomorphism and to prove or disprove specific groups were isomorphic. Later studies on isomorphism focused on developing local instructional theories to inform teaching isomorphism. In 2009, Larsen recorded a teaching experiment in which participants were expected to generate a definition of isomorphism. Later, Larsen et al. (2013) noted that the homomorphism property was more challenging for students to unpack than the bijection property. Additionally, Larsen (2013) noted, "students' use of the homomorphism property is usually largely or completely implicit" (p. 722).

Recently, Hausberger (2017) addressed students' understanding of both isomorphism and homomorphism through a textbook analysis and teaching experiment in which he observed the failure of textbooks to define "structure" in the context of "structure-preserving" isomorphisms and homomorphisms. Thus although some work on students' understanding of isomorphism has been addressed, such as the focus on sameness in naïve isomorphism, limited attention has been paid to students' use of language or images while considering homomorphism and teachers' conceptions of isomorphism and homomorphism have been ignored.

Methods

Participants included two faculty members and two students from each teacher's junior-level abstract algebra class. Both teachers had taught the course at least once before. Instructor A was tenure-track faculty, and Instructor B was a full-time instructor. The students' backgrounds varied; all had mathematics as at least one major and had previously taken an introduction to proof course, but some were double majors and other previous coursework varied. Teachers were recruited at the beginning of the semester from that semester's abstract algebra teachers. Students were recruited based on their responses to a survey as part of a wider project.

Data for this paper are drawn from classroom video and a round of interviews with students and teachers. The classroom video data was collected from days when isomorphism or homomorphism-related topics were discussed in class. Participants engaged in semi-structured interviews (Fylan, 2005) lasting roughly one hour each. The relevant interview questions focused on definitions, descriptions, and explanations for a 10-year-old of the concepts of isomorphism and homomorphism. Interviews with teachers occurred as they began teaching isomorphism. (Both taught isomorphism before homomorphism.) Interviews with students occurred after their class learned about the FHT and took an exam on group isomorphisms and homomorphisms. All interviews were audio and video recorded and any written work was collected.

The interviews were transcribed and coded using thematic analysis (Braun & Clarke, 2006). This included multiple iterations of coding (Anfara, Brown, & Mangione, 2002); first, transcripts were open-coded for vivid, active words that could indicate conceptual metaphors; next, statements were viewed holistically for mathematical approaches being conveyed by statements; finally, codes were generated and refined by repeating the previous stages. These codes were influenced by Hausberger's (2017) ideas of structuralism and Zandieh et al.'s (2016) work with functions; specifically the Input/Output, Morphing, and Traveling codes are similar to the latter's definitions. The codes generated from this process are given and defined in Table 1. The classroom video was selectively transcribed; segments when isomorphism and homomorphism were originally defined and when the FHT was introduced were completely transcribed. However, technical proofs or computations and difficult to hear segments were excluded. The classroom transcripts thus generated were coded like the interview data.

Table 1. Codes, descriptions, and examples.

Code	Description	Common Examples
Embedded	Structure inside a structure	"living inside"
Input/Output	Function machine language where entry leads to new result	"spit out," "pop out"
Matching	Elements or structures aligned	"match," "line up," "correspond"
Morphing	Elements or structures altered from original format	"collapse," "condense," "transform"
Relabeling	Names of elements rearranged	"relabeling," "renaming"
Sameness	Structures equivalent in some way	"same exact thing," "equivalent structures"
Sight	Visual imagery used	"reflected," "image"
Structuralism	Structure-based language of the formal definition	"operation-preserving," "structure-preserving"
Traveling	Element or structure moves from location to location	"from G to H," "go to," "send to," "hit"

Results and Discussion

Metaphors in Instructor Interviews

Class A. Instructor A used a variety of language to address isomorphism, including structuralism ("preserves the operation") and traveling metaphors (e.g. "a function from one group to another group"). However, most of her discussion of isomorphism centered on two metaphors: renaming and sameness. She seemed to view renaming as more indicative of isomorphism (the function) and sameness as indicative of groups being isomorphic:

> So if I was trying to explain isomorphic…I would say two things are the same, just with different names. If I was trying to find…[an] isomorphism, I'd say it was…how I decided to rename the things in one group as the things in another group.

Instructor A initially described homomorphism using structuralism, saying it was "a mapping that preserves operation." Later descriptions used mostly sameness, traveling, and morphing language, often in conjunction with each other as she structured her thoughts around the FHT:

> So this is my domain and let's say there's a bunch of elements in here….My homomorphism clumps them into like regions or sets. So this is kind of all working inside my domain, and then I have my function that goes over to my range, and now this set is sent to a single element over here.…The operation between these sets is the same as operation between those elements.

Class B. Instructor B used a variety of metaphors to discuss isomorphism, including matching, relabeling, and sameness, in addition to structuralism. When discussing isomorphism as a function, he used language like a "relabeling of elements," a "correspondence that matches like things with like things," and a "mapping between two algebraic structures that preserves the structure." Common language for isomorphic groups included talking about "equivalent structures" or "there's really no difference between these structures," where "structures" meant algebraic structures like groups or rings. His preferred view of isomorphism was as a relabeling:

> From an algebraic point of view, there's really no difference between these structures, and so…if you just took these elements and attached these other labels instead of the labels you originally had, you get the same exact structure. So that's the idea I try to get across more than… a bijective function that…preserves such and such operation. So I think it's really the relabeling is the most natural way to think of it.

When discussing homomorphism, he initially used structuralism and traveling language (a "map from one structure to another structure"). However, he later used more sight and sameness language to contrast with isomorphism: we "kind of don't really initially see how the…structure within the…domain group is reflected in the…codomain whereas with isomorphism we…see that right away. Right we just see that it's…equivalence of structures." When pressed, he gave a more vivid picture of homomorphism that included morphing, sameness, and traveling language:

> I guess you could sort of view it as threads condensing into a single…element in the codomain and…then those would become equivalence classes modulo the kernel of…the map etc. etc. If we look at the…7 elements that get mapped to a particular element, then what we really have is this, this equivalence class modulo the kernel, and…if we mod out by the kernel then we can take any one of those things as a…representative.

Metaphors in Instruction

Class A. Instructor A used inquiry-oriented materials based on the local instructional theories developed by Larsen and colleagues (e.g. Larsen, 2013; Larsen et al., 2013) to have students reinvent the definition of isomorphism in class. This is significant because it meant her students talked about isomorphism before a definition was given. Pre-definition, most public language describing attempts to map between a mystery table of six elements and D_6 (dihedral group of six elements) was matching metaphors. For example, consider the following exchange:

> *Student:* I think it's harder to find what each element corresponds with the letter because they're self identities, but the ones that are not self-identities are D and G so it's easier to see which ones…are the only two elements that are not self-identities.
>
> *Instructor A:* Right, so this is the game you're playing, you're trying to correspond these letters with D_6 elements?

However, when a definition was given, the language Instructor A used largely matched what she had said in her interview, while also incorporating the matching language the class had used:

> These correspondences we have been working with are potential *isomorphisms* that allow us to "rename" elements in G with elements in H and then verify the operation to show that G and H are essentially the same.

She introduced the homomorphism definition before teaching the FHT. Pre-theorem, most homomorphism-related discussion was based on the formal definition or traveling metaphors (e.g. "What's something that definitely gets sent to the identity?"). However, to describe the FHT, Instructor A utilized the imagery from her interview:

> We can use the homomorphism to…construct bands where all of these little elements get sent to the same thing so they're grouped together. And what they're grouped together into are their elements in the quotient group.…So all the little dots that get sent to x will form a coset in our quotient group….And the partitioning we would have under the quotient group is the same that we'd have under the homomorphism.

Although she used less vivid language to describe the theorem in class, her explanation did include elements of sameness (final sentence) and morphing (grouping) as in her interview.

Class B. Instructor B used similar language in class to describe isomorphism as he had in his interview. He mainly utilized traveling (e.g. "identity kind of has to go to identity") and matching (e.g. "what would have to go with what") language when discussing how to approach specific mappings with students. He also referred to isomorphism as "the rule that's doing the relabeling," utilized structuralism in a manner similar to his interview, and extensively referred to isomorphism as "essentially the same" or when "two groups have exactly the same structure."

Instructor B's metaphors for homomorphism in class differed from his interview metaphors to a large extent. In class, he frequently spoke of homomorphism as a function using traveling metaphors (e.g. "So homomorphism is essentially a map, and again this could be from structure to structure in general. In our case it's from one group to another group…"). He also used structuralism on a number of occasions (e.g. "…and so it is a map that preserves whatever operation we have, in this case the group operation, but is not necessarily a bijection."). Although he drew on morphing language to describe homomorphism in his interview, his description in class drew more on an embedding metaphor when first discussing the FHT:

> The way to think about this then is if you've got a surjective homomorphism, then the range H essentially is already living inside of G somehow. All the information about H is already here, and in fact we can recover H purely in terms of G by taking the factor group of G mod the kernel. So we get an isomorphic group where we don't even have to refer to H at all. It's just purely in terms of G.

Student Metaphors

Class A. The majority of the language used by both students from Class A focused exclusively on the formal definitions. For example, Student 1A described an isomorphism as:

> …basically a function that maps one group to another group such that the function is one-to-one and onto and such that the function of the combination of two values in the first group is equal to the function of the first value combined with the function of the second value.

However, he moved beyond the definition to sameness when asked how to describe an isomorphism to a 10-year-old: "If two groups of numbers or anything are the same." However, the idea of "sameness" seemed to confuse him as well. When he was trying to describe a homomorphism for a 10-year-old, he noted, "…when you explain that it's two groups don't have to be the same then it gets really confusing on what is a homomorphism and what isn't a homomorphism." In trying to distinguish between isomorphism and homomorphism, he seemed unsure how to take bijection away from isomorphism and still have a coherent mental picture.

In addition to the formal definition, Student 2A used matching and sameness metaphors for isomorphism. For instance, he used two circles of ten colored marbles in a matching metaphor:

> …then you number them also 1 through 10 but instead you…rotate it so you don't have 1's matching up with the 1's and… so the 1 in the red matches up with the 3 in the blue, and then…you figure out if you have 1 plus 3, that'll get you to marble 4. Well marble 4 matches to marble 6 or whatever, so something like that.

His example about work being independent of path emphasized sameness: "The idea is…regardless of how you go, it's the same ending spot, so what you're doing is actually the same operation; this just looks different."

He expressed ideas like "isomorphism is a fancy case of homomorphism" multiple times and did not make much effort to distinguish between isomorphism and homomorphism. When pressed on homomorphism, he returned to the marble example, noting this time you could have less marbles "and…now you're allowed to overlap." He maintained the matching metaphor across isomorphism and homomorphism, but did not retain the sameness metaphor.

Class B. Student 1B's isomorphism language aligned with Instructor B's to a large extent as he coordinated sameness, relabeling, and structuralism language: "I guess an isomorphism would be a function, which is bijective and it's structure-preserving…I mean… basically, you can just relabel the Cayley table, but that's formalized as f of ab equals f of a times f of b."

Student 1B's language for homomorphism drew on metaphors and the FHT like Instructor B:

> A homomorphism is just a function that preserves the structure…not necessarily all of the structures; it might just preserve one structure. Like the integers map to Z mod 2 or something, that could preserve the structure of like the evens and the odds, but it destroys a lot of the other properties of the integers….[Preserving the structure] would be that definition: that f of a product b equals f of a product f of b, but…it's intuitive for me to go back and think about the Cayley tables because they're just saying that wherever the product of these two things gets mapped to gets mapped to wherever the product of wherever these two other things gets mapped to, so…that's the structure right there that's being preserved: things still will be nice and well-defined and play nicely…..

Notice he used traveling language as he described the integers mapping to Z mod 2, much like Instructor B's use of traveling language. He utilized structuralism through preserving the structure. However, he seemed to use the word "structure" in two senses: the homomorphism definition and an imposition of order. This latter sense is similar to Instructor B's embedding description of the FHT given in class, in which the emphasis was on the structure of the domain.

Student 2B defined isomorphism as, "an operation through which you would transform an element of one group to the corresponding element in an identical group," which utilized morphing, matching and sameness metaphors. He also gave a vivid sight metaphor coordinated with sameness language when asked what he would say to a 10-year-old:

> …isomorphism is, is closer to the mirror….Like you get the same thing back….But you look in, just like a regular mirror straight on, it's pretty much the exact same thing back, but it's not you. It's just an image of you that retains all the characteristics.

Student 2B's language for homomorphism was in many ways similar to his language for isomorphism. His initial description coordinated morphing and sameness metaphors: "an operation through which you would transform an element in one group to a group with similar characteristics that is of lesser or equal size." When later prompted about how he would describe homomorphism to a 10-year-old, he again shared vivid metaphors. He expanded on the sight-based mirror imagery from isomorphism to compare and contrast with homomorphism: "…sometimes you have mirrors that make you look smaller like at the corners of hallways and hospitals. Sometimes it's a little bit smaller. That's like a homomorphism." He also gave a morphing metaphor: "Look at your dad and then look at yourself. Imagine…what part of your dad went to you sort of as a homomorphism….he took a part of himself and sort of condensed it to create you.…" Although he used a condensing image like Instructor B's interview response, his condensing image did not possess the clear FHT structure of Instructor B's response.

Discussion

Returning to research question 1, the teachers were largely consistent in their metaphor usage in the interview setting and in class. Both teachers focused on sameness (more for isomorphic structures) and renaming/relabeling (more for the isomorphism function). Both also relied on the FHT and morphing, sameness, and traveling metaphors to provide meaning for homomorphism beyond the formal definition. However, they structured their understanding around the FHT differently: Instructor A focused on morphing within the domain and then traveling to produce sameness between the groups whereas Instructor B morphed while traveling to produce sameness (interview) or viewed the relevant sameness as being embedded in the domain (in class).

Addressing research question 2, there was some alignment between metaphors used in class and metaphors used by students. All four students utilized sameness language for isomorphism like had been used in class, though Student 2A also used a lot of matching language and Student 2B incorporated morphing language for isomorphism. However, their images of homomorphism varied widely. Students 1A and 2A did not use sameness to describe homomorphism. Student 1A seemed to try separating isomorphism and homomorphism by removing sameness to reach homomorphism, but did not know where that left him. Student 2A used matching metaphors for both isomorphism and homomorphism but only applied sameness to isomorphism. Neither student from Class A used an FHT-based picture like their teacher had used, though it is possible that Student 2A's matching language was based on pre-FHT discussion around homomorphism. Students 1B and 2B had more distinct images for homomorphism and were closer to aligning with their teacher. Student 2B used condensing language to describe homomorphism, though he did not give evidence of attention to structure within the group being condensed. Student 1B was more aligned with his instructor's embedding view from class, based on his attention to some type of organization being highlighted and shared between the domain and codomain.

Conclusion

Isomorphism and homomorphism are concepts central to the study of mathematical structures, specifically within abstract algebra and in math more broadly. Thus deepening our understanding of how teachers and students think about these concepts and what conceptions are communicated from teachers to students is critical. In this study, the naïve isomorphism view of sameness (Leron et al., 1995) was broadly shared whereas the images of the FHT used by instructors were not broadly shared, and the images used by students varied widely. These varied metaphors revealed varied conceptions (e.g. elements traveling to elements, shared structure inside groups, transforming from group to group) that may be more or less useful when solving problems. Thus future work includes investigating what isomorphism and homomorphism problems students with these metaphors can solve, especially because most descriptions given by the students aligned (to some extent) with the definition. Furthermore, other teachers and algebraists may or may not share these teachers' FHT-based images of homomorphism, so ascertaining other expert views of homomorphism is essential for future study.

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