Using a Dynamic Geometric Context to Support Students' Constructions of Variables

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Using Thompson and Carlson's (2017) definition of a variable and the results of teaching sessions with two preservice secondary mathematics students, I describe the role of quantitative and covariational reasoning in constructing a formula with variables to describe a relationship between covarying quantities in a dynamic geometric context—the Parallelogram Problem. I report that although each student reasoned with a dynamic situation, their symbolic representations of that situation did not necessarily entail variables. I conclude that providing students with dynamic situations with which to construct formulas provides them opportunities to construct formulas with variables representing covariational relationships between quantities.

Keywords: Cognition, Precalculus, Preservice Secondary Teachers

Dreyfus (as cited in Izsák, 2000) claimed, "There must be some meaning association with a notion before a symbol for that notion can possibly be of any use" (1991, p. 31). At the time Küchemann (1981) had identified some different ways in which children interpreted "letters" (p. 110), but the pervasive difficulty of meaningful symbolization meant that he and other researchers continued to focus on students' understanding and construction of meaningful symbols (e.g., Izsák, 2000, 2003; Kaput, 1992; Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Stephens, Ellis, Blanton, & Brizuela, 2017; Thompson, 1990, 1994b; Trigueros & Ursini, 2008). In this paper, I focus specifically on students' conceptions of symbols as variables within formulas. Thus, rather than students interpreting symbols as static unknowns (Dubinsky, 1991) or fixed, given referents (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), my goal for my teaching sessions (Steffe & Thompson, 2000) with preservice teachers was for them to conceptualize a symbol in a formula as representing a quantity whose value changes within a dynamic situation (Thompson & Carlson, 2017). I provide insights on how covariational reasoning—in which students conceive of situations as composed of quantities that vary in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002)—influenced students' construction of variables in formulas. I note that Thompson and Carlson (2017) argued that students' images of covarying quantities can differ dramatically, and moreover, that reasoning with a dynamic situation does not necessarily imply that a student conceives of smooth variation (see Castillo-Garsow, Johnson, & Moore, 2013). Both of these ideas meant that although a dynamic situation supported students' reasoning about variables, it alone was not sufficient for a student to construct a variable. Thus, I highlight the importance of having students attend to the roles of variables in representing covariational relationships, an important topic in calculus ideas (Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, 2017).

Background

Images and Theoretical Perspective

I adopt the radical constructivist perspective (von Glasersfeld, 1995) that individuals actively construct quantities and that an individual's image of a situation is projected from their mental organization of sensory data. The notion of image I am referring to stems from Piaget's (1967) descriptions of images as shaped by mental operations individuals perform. Thompson described the implications of this perspective on individual's viable images by noting while "the image is shaped by the operations, the operations are constrained by the image, for the image contains vestiges of having operated, and hence results of operating must be consistent with the transformations of the image" (Thompson, 1996). Moore and Carlson (2012) explored this relationship between images and operations when researching the role of images in the construction of a formula for the volume and height of a box. Travis's image of the situation differed from the image the researcher intended, but his resulting formula accurately represented his image of the situation. From this study, Thompson and Carlson (2017, p. 448) reemphasized the idea that students' constructions of symbolic expressions are constrained by the quantitative structures they construct about a situation. This idea is important to the notion of variables I use here because I argue students' images of a situation should be compatible with their formula.

Distinguishing Between Quantitative and Numerical Operations

Researchers have advocated for students' thinking of symbols as variables by involving a conception of varying values (e.g., Janvier, 1996; Kaput, 1994; Küchemann, 1978; Trigueros & Ursini, 1999). In an effort to support students' conceptions of a variable, I emphasize the role that differentiating between quantitative and numerical operations played in distinguishing between students' conceptions of variables/formulas and their images of a situation. Thompson (1994) summarizes this difference: "A quantitative operation is non-numerical; it has to do with the *comprehension* of a situation. Numerical operations are used to evaluate a quantity" (1994a, pp. 187-188, emphasis in original); a quantity is a measurable attribute (Thompson & Carlson, 2017). Thus, when I am referring to students reasoning quantitatively with a formula, I am referring to one of two cases. First, I refer to a student connecting the quantitative operations they have in a situation with the symbols in their formulas that represent numerical operations such that the symbols are conceptually tied to those quantitative operations. Second, I refer to a student constructing a formula from a given situation by considering the quantitative operations involved in relating the quantities with the anticipation that the formula also represents corresponding numerical operations.

Variables and Formulas

Symbols can serve different purposes to students depending on the meanings they attribute to them. Because of the focus on quantitative reasoning in regards to formulas, I rely on Thompson and Carlson's (2017, p. 425) three different meanings for symbols: constant, parameter, and variable. A person constructs a *constant* if the person envisions a quantity as having a value that does not vary. The symbol can take on different values, but these values do not change as the result of an image of variation. A person constructs a *parameter* if the person envisions the quantity as having a value that can change from setting to setting but does not vary within a setting. A person constructs a *variable* if the person envisions that a quantity's value varies within a setting. Unlike other researchers' approaches to variable conceptions, this study does not examine the construction of variables within a graphical setting (cf. Chazan, 2000), nor is it a more general description of a letter taking on different values (cf. Blanton, Levi, Crites, & Dougherty, 2011; Blanton et al., 2015; Izsák, 2003) or an attempt to identify non-quantitative meanings for "letters" (cf. Küchemann, 1981). This study does build on the work of others who are using contextual situations as a means to construct formulas at the elementary (Panorkou, 2017) and middle school levels (Matthews & Ellis, in press). The former reported successful covariational reasoning with a dynamic rectangular area context and the latter identified students' difficulties with reasoning about rates of change with the dynamic situation.

Methods

In an effort to understand the mental operations involved with constructing formulas through covariational reasoning, I conducted a study with four preservice teachers (two of which I report on here) from a large public university in the southeastern U.S. These participants were successful mathematics students (passed at least two upper level mathematics courses and Calculus sequence) who had experience with thinking critically about secondary and post-secondary mathematical ideas through their coursework at the university. Thus, they afford insights into how students with vast mathematical experiences conceptualize variables given dynamic situations. These students had just completed their first or second semester in a four-semester secondary mathematics education program during which they completed a secondary mathematics topics course designed from the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015). The study consisted of 3-5 exploratory teaching interviews (Steffe & Thompson, 2000) lasting 1.5-2 hours each over the course of four weeks. Each student answered the same sequence of pre-designed tasks. However, I encouraged the students to think aloud (Goldin, 2000) and I asked questions based on my understanding of their activity. The goal of my questioning was to build viable models of the *students' mathematics* (Steffe & Thompson, 2000). I conducted open (generative) and axial (convergent) analyses (Strauss & Corbin, 1998) to inform these second order models of the students' construction of formulas.

Task Description: The Parallelogram Problem

In this section, I describe some of the mental operations I hypothesize are involved in constructing a formula to represent the relevant covariational relationship. This description will provide insights into how to conceive of variables through covariational reasoning to construct formulas in the context of a novel situation. It is important to note that this description also involves constructing constants and a parameter as symbols for a formula, but my focus here is on how this dynamic situation supports the construction of variables.

The following is a description of the Parallelogram Problem, in which I presented students with the manipulative in Figure 1a and the following prompt: "Describe the relationship between the area inside the shape (shape formed by two pairs of parallel lines) and one of the interior angles of the parallelogram (up to a straight angle)." In a traditional construction of the formula to describe this relationship, a student need only work with a single static figure of a parallelogram, constructing *constants* in the situation and relating these constants to produce a formula; no images of variation are necessary to accomplish this goal beyond understanding that different states of the figure might correspond to different values. Although there is an underlying assumption that these symbols can take on different values, this construction of symbols in a formula do not fit with the notion of variable defined earlier. The goal of giving the students a shape they could manipulate was to support their construction of a variable by having them consider the covariational relationship of the area of the parallelogram *ABCD* and the openness of $\angle DAB$.

In order to motivate the construction of a variable for a formula via covariational reasoning, one needs to quantify two quantities in a situation and then determine the covariational relationship between them. One potential first step for the Parallelogram Problem is for a student to construct the relationship between the area of a parallelogram and a corresponding rectangle. A student can conceive a parallelogram's area as equivalent to the area of the rectangle constructed by translating a triangular region ($\triangle DEC$) alongside base $\overline{AD}$ (Figure 1d/e). Then, a student can anticipate that as $\angle DAB$'s openness increases, a rectangle with equivalent area can be produced for each instantiation of $\angle DAB$'s openness. At this point the student has determined the two quantities to covary: the angle of the parallelogram and the area of the rectangle constructed from translating the triangle shape in the parallelogram (Figure 1f). Reasoning about amounts of change, a student can then consider equal changes in $\angle DAB$'s openness and attend to the corresponding areas and changes in areas of the rectangle (and, equivalently, the area of the parallelogram) to make, for example, the following conclusion: For equal changes in $m\measuredangle DAB$ from 0 to π/2 radians, the measure of the area of the parallelogram is increasing by decreasing amounts with respect to angle measure (see shaded areas in Figure 1f).

To construct a formula with variables from this situation, a student can use a meaning for the sine relationship that involves understanding it as the height above the center of a circle measured in radii (Moore & LaForest, 2014). That is, the length of $\overline{AB}$ rotated around leaves the traces of a circle centered at point *A* with radius $\overline{AB}$ (Figure 1i), a non-trivial connection (Hardison, Stevens, Lee, & Moore, 2017). This connection between the covariational relationship identified in the situation and the sine relationship is what enables the construction of variables within a formula. That is, a student can represent the covariational relationship between the height of a parallelogram in the situation and $m\measuredangle DAB$ with the sine relationship (Figure 1g/h). Moreover, the student can use of symbols for the sine relationship (which entail numerical operations) to represent the covariational relationships identified in the situation. This reasoning is the second way of quantitative reasoning with formulas described previously.

From there, a student can construct a relationship between quantities to produce $EB = \sin(\theta)$, where $\sin(\theta)$, in this formula, represents the height's changing value in the situation. This magnitude is multiplied by the value of the length of the base of the parallelogram, *AD*. Thus, the final formula for the area of the parallelogram is $Area = AD\sin(\theta)$. Here, based on the student's image of the situation, the variables are *Area* and θ. I will not include a discussion of the role of units in this formula here, and assume that the student is constructing an area measured in *radii*^{2}. However, see Alexandria's example in the results section to see how this image can be extended to consider other units, thus resulting in a normative formula.

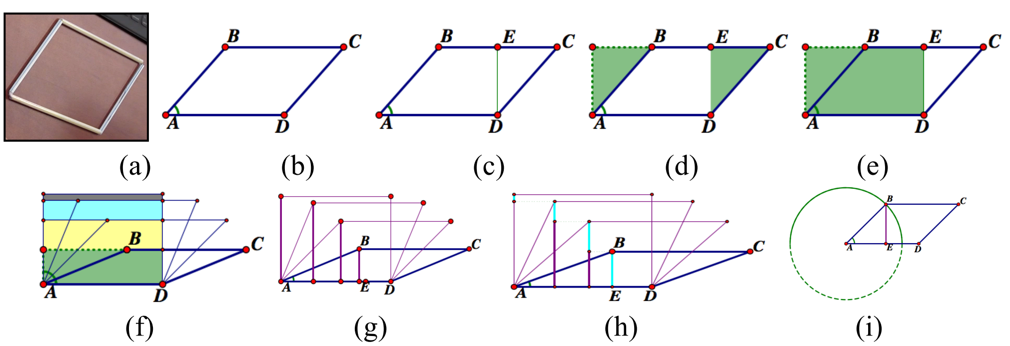


Figure 1. Image from Stevens (in press) (a) manipulative with changeable angles (b) labeled parallelogram (c) labeled height DE (d) triangular region in parallelogram translated to form rectangle (e) rectangular region with equivalent area to parallelogram (f) various colored areas indicate amounts of change in area for equal changes in angle measure (g) dark purple segments indicate various heights for equal changes in angle measure (h) light blue segments indicate amounts of change in height for equal changes in angle measure (i) height of the segment above $\overline{AD}$ *as the fractional amount of the radius AB of a circle centered at A (i.e., the sine relationship).*

Results

I now describe students' different types of formula construction as it relates to variables for the area of the parallelogram based on my analysis of Charlotte and Alexandria's interviews. To offer an indication of the difficulties the students initially had with the Parallelogram problem, I note that three of the four students (including Charlotte and Alexandria) initially attempted to justify that the area remained constant as they manipulated the object. When pushed on their justifications, they began to doubt their initial claims and were motivated to attend to the quantities to form new justifications. The remainder of the section focuses on the results of this reasoning as it relates to their formulas.

Constructing Multiple Systems of Measurement with Constants to Describe One Situation

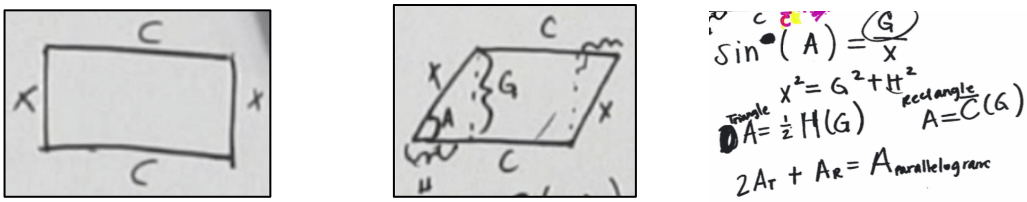
Before attempting to reason covariationally about the quantities, Charlotte had constructed a sequence of calculations (Figure 2c) to carry out in order to determine the measurement for the area of a specific parallelogram. This sequence was the result of reasoning with the static parallelogram in Figure 2b. This process was similar to the description for a traditional construction of the formula in that each of her symbols represented a constant from a static figure she drew. The process differed in that she did not combine all her sequences of actions to calculate the measurement into *one* formula.

Her goal for constructing this sequence of calculations was to compare its resulting value to the value for the area when the shape was a rectangle. She knew that to calculate a value for the area of a rectangle she should multiply the length of its base with the length of its height. She wanted to calculate these two measurements "because then I could compare-like I wouldn't be-I wouldn't be assuming based on like my eye, like changing- like trying to figure out how the area changed. I would know like-I would have concrete numbers." That is, she wanted to make gross comparisons between the numerical values to determine if the two areas' values were different.

Charlotte wanted to calculate the measure of the area for the shape when it was a rectangle by multiplying X and C together (Figure 2a). She then wanted to determine the measure of the area of a parallelogram with a given angle measure (Figure 2b) using her sequence of calculations (which would have resulted in the correct area measurement) (Figure 2c) and compare the resulting values. It is important to note that although this latter sequence of calculations could actually be used to find the area of every parallelogram in the situation, she viewed each of the symbols in her formula as unknown constants for that specific instance of the parallelogram. For instance, she referred to the angle as "the angle that I picked" and that she "would know the value of the angle" when she went to calculate the measure of the parallelogram's area with that angle.

The constructions of her formula and sequence of calculations themselves were insufficient for her to make a conclusion about whether or not the area of the parallelogram changes because she did not know what values to use. She realized this issue only after constructing her two systems of measurement, stating, "Okay. So, I don't know. If I-If I do algebra then I could see-I could tell you maybe." However, she did not continue trying to relate the two systems. In fact, by the end of her attempt with this strategy of comparing the two instances, she still anticipated that the areas for each would probably have the same value.

This conception of formulas differs from the one described in the first order model because she constructed two different systems of measurement to describe one situation. Thus, even though she identified changing angle measures and wanted to make conclusions about the directional covariational relationship between angle measure and area in the situation, neither her formula nor her sequence of calculations were the result of covariational reasoning. Rather, they were the result of analyzing static figures. Thus, all of her symbols represented constants.



(a) (b) (c)

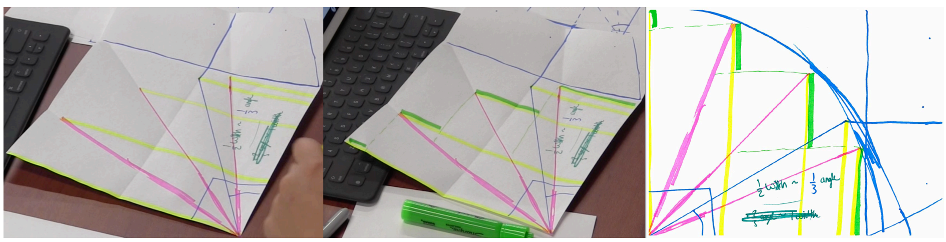
Figure 2. (a,b) The two instances of the situation Charlotte whose areas she wanted to measure and (c) her sequence of calculations in order to measure the area of the parallelogram in Figure 2b.

Constructing a Formula Disconnected from the Students' Image of the Situation

Later in the interview, Charlotte shifted to a different approach and went through steps similar to those described in Figure 1b-e. She used the manipulative in Figure 1a to describe how she wanted to calculate the area of the parallelogram. She wanted to multiply the "distance between yellow [side length] to yellow [side length]" in the manipulative by the "length of the yellow [side length]". Even with this new insight, Charlotte remained unsure whether the area changed. She concluded, "I reckon. It's really hard to tell because like in the back of my mind, I think, 'Oh, just because it's getting narrower-more narrow, I'll say it like that, it doesn't mean necessarily that the area is decreasing." At this point, Charlotte had a new way to calculate the area of the parallelogram by reasoning quantitatively about the situation. However, her image of the situation made it difficult to reconcile whether or not this new method was an appropriate way for her to reason about the directional covariational relationship between the quantities. Her uncertainty demonstrates that although she had constructed a way to measure area that involved her being able to make comparisons between different states using the same formula, the numerical operations she noticed that resulted from her new formula were not entirely connected to how she conceived of the quantities in the situation. One hypothesis for this uncertainty is that she had constructed this new calculation for the measurement by focusing on instances of the parallelogram to rectangle translation instead of imagining a smooth image of a growing rectangle she conceived. Regardless, the disconnect between her conclusions about the situation and her meaning for the symbols in her formula (i.e., the two distances between the side lengths) make it so that the latter do not fit with the aforementioned definition of a variable.

Constructing a Formula that Entails Students' Identified Covariational Relationships

Alexandria, similar to Charlotte, had gone through the process outlined in Figure 1b-e. At this point, Alexandria thought there was a linear relationship between the angle measure and the height of the parallelogram (which to her also implied a linear relationship between the angle measure and area of the parallelogram). To check, Alexandria constructed four equal changes in angle measure. She constructed the pink side lengths of the parallelogram (Figure 3a) and then, in yellow (Figure 3a), marked the corresponding heights for each marked angle measure. She then highlighted in green (Figure 3b) the segments that represented the change in height for each successive equal change in angle measure from a "full" right angle downwards. She claimed, "The change in height increases," and then concluded, "It's not linear." At this point, Alexandria had reasoned covariationally about the quantities in the situation, but she did not have a formula. Alexandria stared at her work for about 25 additional seconds, and then muttered, "Don't tell me this is sine?" She drew in a quarter circle in blue (Figure 3c). She then began to describe how she drew in her blue curve based on the pink lines, and then suddenly exclaimed, "Oh, it's a radius! No, duh. Ding ding ding ding ding! The pink line's a radius." Thus, Alexandria's meaning for the sine relationship entailed a covariational relationship between an angle measure and a height quantity that she was also able to identify in this situation (as outlined in Figure 1g-i). After this point, Alexandria quickly constructed a final formula to represent the relationship between the angle measure and area of the dynamic parallelogram. She wrote the following formula: "$L\sin(\theta)=Area$" and "L=AD", calling L her "length" and pointing to $\sin(\theta)$ saying that it "gives me my height." When I asked her what units she was using, she almost immediately wrote, "$Lr\sin(\theta)$" saying that $\sin(\theta)$ "alone [*circles sin(θ)*] gives us how much of a radius it is, so how-how much of this whole [*pointing along the edge of her paper corresponding to the height of the parallelogram with right angles*]...so I put the r [*pointing to the r in her formula*] in there to give me a typical measurement that we're used to." Thus, in this situation, she conceived of both angle measure and height changing in the situation, and she constructed the variables θ and Area to describe her image of how quantities are varying within a dynamic situation.



(a) *(b)* *(c)*

Figure 3. Alexandria's (a) construction of heights (yellow) (b) construction of amounts of change in height (green) and (c) construction of a quarter circle (blue).

Discussion and Conclusions

I proposed two ways in which a student could reason quantitatively about a formula. The first involves connecting quantitative operations with numerical operations. Charlotte's activity demonstrated the difficulty of this reasoning. Her formulas were the result of quantitative operations using static images of the situation, which she then wanted to use to reason about the covariational relationship between quantities. However, in the first case, she was unable to use her two systems of measurement to draw a conclusion, and in the second case, she struggled to reconcile her image of the situation with the numerical operations resulting from her quantitative reasoning with the situation. These disconnects illustrate how important it is for a student to be able connect quantitative operations within a situation with the symbols that represent not only quantitative relationships but also numerical operations for measurement. Alternatively, Alexandria demonstrated the second way of reasoning quantitatively with formulas because she was able to identify a covariational relationship within the situation and construct variables, and ultimately a formula, that represented both the quantitative and numerical operations she conceived. Her reasoning illustrated a powerful conception of a variable in that it enabled her to construct a formula, which, unlike Charlotte's formulas, represented a covariational relationship between quantities. As a result of this study, I conclude that students should be provided with more opportunities to construct variables via reasoning covariationally with dynamic situations.

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